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A Discrete Density-Dependent Model of the Solanum Virus

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By

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Abstract

Compartmental modeling has been used to model infectious diseases for roughly 100 years. Since 2009, several papers have modeled zombie outbreak using this method with various results. This paper will develop a unique model for the spread of the *The Walking Dead* zombie virus throughout the contiguous United States. Frequency dependent and density dependent transmission will be discussed, and density dependent transmission will be shown to be the appropriate choice for this model. Constant parameters, such as birth rate, bite rate, death rate, and turning rate will be determined using real-world and fictional data. After developing a basic model, modifications will be made to include the airborne pathogen and latency. A system of autonomous differential equations can then be derived, followed by a dynamical system of difference equations. The system of differential equations will be used to determine equilibria, if any exist, stability will be investigated, and numerical solutions will be calculated. Pithing, a practice usually reserved for slaughtering livestock, will be considered for human use and shown to help control the zombie population. Numerical solutions will show that pithing alone will not save the human race, but used in conjunction with zombie removal by a well-organized force, a successful zombie removal rate can be established. Finally, the model will then be modified to include zombie removal calculations and vaccination. The goal of this paper is to develop a zombie model that represents AMC's *The Walking Dead* outbreak and develop numerical methods by which mankind can calculate appropriate actions.

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Variables and Parameters for the SEZ and $SEZV$ Models

$S =$ number of individuals susceptible

$E =$ number of individuals expired

$Z =$ number of individuals zombified

$V =$ number of individuals vaccinated

$\alpha = \%$ of E not pithed

$1 - \alpha = \%$ of E pithed

$\beta = \%$ of S bitten per zombie per hour

$\gamma = \%$ of Z removed per susceptible per hour

$\delta = \%$ of non-bite related death for S and V per hour

$\zeta = \%$ of S vaccinated per hour

$\lambda = \%$ of E turned per hour

$\pi = \%$ of S and V born per hour

$\chi = \%$ of pithed removed from E per hour

1 Introduction

> The dead walk among us. Zombies, ghouls — no matter what their label — these somnambulists are the greatest threat to humanity, other than humanity itself. To call them predators and us prey would be inaccurate. They are a plague, and the human race their host.
>
> -*The Zombie Survival Guide*, Max Brooks

Infectious disease has captured the imagination of writers and directors over the past 50 years and has fascinated scientists for centuries. No other infectious disease has been portrayed in television and movies more than the zombie virus. According to Brooks, a zombie is an animated corpse that feeds on living human flesh [1]. Books, comics, movies and television shows have created many types of zombies for entertainment purposes, and few have been more embraced by the general public as those in AMC's *The Walking Dead*. Like most zombie stories, this show follows a handful of survivors as they fight through the zombie apocalypse.

A zombie-like virus may be more of a possibility than most people realize. Polar ice caps and ancient glaciers are receding every year, exposing parts of the earth that have not see daylight for millennia. According to Morelle, a virus, Pithovirus sibericum, was recently retrieved by French scientists from the Siberian permafrost [4]. Although this virus is harmless to plants and animals, after 30,000 years of entrapment below 30 meters of frozen earth, the virus is able to infect living organisms. Scientists are worried that more harmful pathogens lie dormant in the permafrost.

The zombie outbreak begins with the release of solanum, the zombie virus, into the general population. This paper will model an outbreak similar to that of *The Walking Dead*. As with most zombie outbreaks, society is quickly thrown into chaos as transmission is possible through direct physical contact of bodily fluids, usually through biting. Unlike most Hollywood zombie outbreaks, this zombie virus is also a dormant airborne pathogen infecting the entire population without detection. Rick, one of the main characters in *The Walking Dead*, learns of this property in the first season's finale [11]. With this greater communicability properties, the solanum virus has infected the entire population of the contiguous United States. The only way to destroy a zombie is to inflict traumatic injury such as bludgeoning, piercing, or burning to the base of the zombie's brain.

The first published mathematical paper on modeling the zombie apocalypse, "When Zombies Attack!: Mathematical Modelling of an Outbreak of Zombie Infection," was authored by Philip Munz, Ioan, Hudea, Joe Imad, and Robert J. Smith, and appeared in *Infectious Disease Modelling Research Progress* in 2009 [8]. A quick internet search reveals the paper is discussed in classrooms at many universities including Texas A&M, Johns Hopkins University, and the U.S. Naval Academy just to name a few. The paper develops a basic model for the zombie outbreak, determines equilibrium and stability through analysis of systems of differential equations, introduces a latency period for the infection, and modifies the model to include quarantine and cure. Next, regular, impulsive reductions in the number of zombies is examined. The discussion within the paper concludes "that only quick, aggressive attacks can stave off the doomsday scenario." Although this paper does model a zombie outbreak, several key characteristics, including human pithing, airborne pathogenetic properties, and possible vaccination, are not included in the model.

In Munz's model, a percentage of zombies that have been removed (their brains have received severe trauma) are allowed to reanimate and continue to infect the susceptible population. This behavior is not consistent with any zombie books, shows, or films. Online course lecture notes can be found from Bryant University, Texas A&M, and Colorado State University that address this issue and suggest alternative models that correct this inconsistency, but these corrections do not model the characteristics of the viral outbreak from *The Walking Dead*.

Other papers, such as "Is It Safe To Go Out Yet? Statistical Inference in a Zombie Outbreak Model," written by Ben Calderhead, Mark Girolami, and Desmond J. Higham, and 'Bayesian Analysis of Epidemics – Zombie, Influenza, and other Diseases," written by Caitlyn Witkowski and Brian Blais, investigate posterior probability distributions for the model parameters. Statistical analysis will not be investigated in this paper, but data from Witkowski's paper will be used to determine a zombie bite rate.

This paper will build a basic model of AMC's *The Walking Dead* zombie outbreak, derive reasonable parameters and introduce a latent period. We then determine any equilibria, stability, and outcomes based on these parameters and discuss methods of zombie removal. Previous papers do not investigate human pithing, vaccination, and the airborne pathogenetic properties of *The Walking Dead* virus. The goal of this paper is to investigate these properties and determine if and how the human race can survive a zombie apocalypse similar to AMC's *The Walking Dead*.

2 The SIR Model

Before building a basic model of a zombie outbreak, let us first observe and discuss the basic SIR model. We first need to lay out the terminology of compartmental modeling, including variables and parameters specific to the SIR model. According to Keeling, an individual who may be infected by some contagious pathogenic agent is called *susceptible*, an individual who has contracted this agent and can spread the infection to others is called *infected*, and an individual who has cleared the infection is called *recovered* [6]. Let S, I, and R be the numbers of susceptibles, infecteds, and recovered, respectively, and let N represent the size of the total population, $N = S + I + R$.

Several basic assumptions must be made before proceeding. We must assume that the population is homogeneously mixed. Initially, we introduce the SIR model in a closed population, that is, no births, deaths, or migration. Hence, N is constant. We assume that encounters between infecteds and susceptibles occur at a rate proportional to their numbers in the population. We also assume that after recovering from infection, the recovered has life-long immunity.

Now we define how the host population, humans, move from and to each of the three compartments, S, I, and R. In the simplest case, in which population size does not change, the only transitions are $S \to I$ and $I \to R$. For the latter, define *infectious period* as the length of time an individual spend in the infected class. The inverse of the infectious period is the *recovery rate*, γ, and γI is the *recovery term*. As noted by Keeling, in more complex models, the recovery rate may vary, but in this paper, this rate will remain a constant [6].

The flow diagram in Figure 1 illustrates the SIR model. The arrows show the movement between the S class and the I class and between the I class and the R class. For transition from S to I, define β as the *infection rate*. Since transmission of the infection depends upon the contact of the susceptible with the infected, the *transmission term* is dependent on two classes, S and I. Thus, the transmission term is βSI. The rate of recovery from infection is dependent only on the number of infecteds, so the recovery term is dependent on I only.

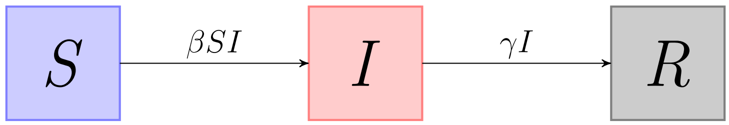


Figure 1: The basic SIR flow diagram

Flow diagrams provide a simple way to determine a system of ordinary differential equations for compartmental models such as the SIR. We follow the arrows in and out of each compartment, adding and subtracting accordingly. For example, βSI will be added to the I compartment, and γI will be subtracted from the I compartment. Using this method, we easily form a system of ODE's:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI\\ \frac{dI}{dt} &= \beta SI - \gamma I\\ \frac{dR}{dt} &= \gamma I.\end{aligned}$$

From this system of ODE's, we form a system of difference equations which we can evaluate numerically:

$$\begin{aligned}S_{i+1} &= S_i - \beta S_i I_i\\ I_{i+1} &= I_i + \beta S_i I_i - \gamma I_i\\ R_{i+1} &= R_i + \gamma I_i.\end{aligned}$$

3 The Basic Zombie Model

3.1 Assumptions

Modeling zombie infection has a few key differences from the SIR model. Let us here be very careful with our definitions, as several terms can be ambiguous in the presence of zombies. Is a zombie 'living' or 'dead', and can a zombie be 'killed'? Does 'kill' imply the loss of life? We will avoid the use of these words in this paper and define a new set of terms that cannot be so easily confused.

We assume that no susceptible individual is immune to the zombie virus. As before, S is defined as the number of susceptible living humans, but I is changed to Z and represents the number of zombies roaming the contiguous United States. The R class no longer represents the number of recovered (as zombies cannot recover from their infection) but instead represents the number of *removed* zombies from the model. Hence, N represents the size of the total 'walking' population, $N = S + Z$. Define β as the bite rate and γ as the zombie removal rate. We also assume that individuals do not enter the Z compartment until the moment they turn into zombies.

3.2 Transmission

In 1995, de Jong published a paper titled "How Does Transmission of Infection Depend on Population Size?" in which a difference between *true* mass action (often called *frequency-dependent transmission* or simply *mass action*) and *pseudo* mass action (or *density-dependent transmission*) is distinguished [3].

> The only difference between true and pseudo mass-action model is in the formulation of the transmission term: for true mass-action the transmission 'constant' depends on N and for pseudo mass-action it is really constant. The dependence of the transmission

constant on N reflects the fact that, whereas for constant density and increasing population size the number of individuals encountered per individual does not change, the probability of encountering any particular individual decreases.

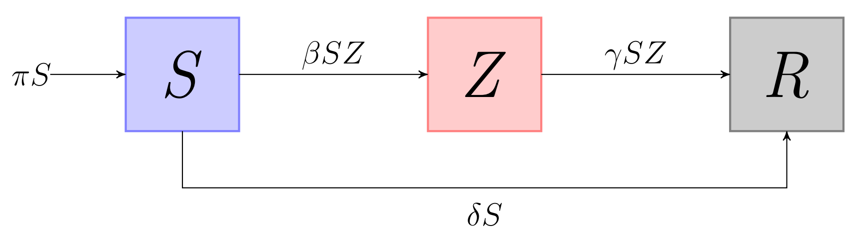
Unfortunately, the terms 'pseudo mass action' and 'true mass action' have created much confusion as many authors refer to either simply as 'mass action.' Abandoning these terms, transmission will be referred to as frequency-dependent or density-dependent.

Modeling of infectious disease over short periods of time allows for very small changes in total population sizes. In these short-term models, N is considered constant, and thus, $\beta SZ = \frac{\beta}{N} SZ$. Hence, there is relatively no difference between frequency and density-dependent distribution over short lengths of time. In longer-term models in which total population changes are taken into account, the difference between density-dependent (βSZ) and frequency-dependent ($\frac{\beta}{N} SZ$) transmission can be significant.

At this point, it will be helpful to determine which type of transmission, frequency-dependent or density-dependent, to employ and disregard the other. In the first season of *The Walking Dead*, civilization has decayed into disorganization. Although the beginnings of the outbreak are not specifically outlined in the show, a shift in population distribution is clear. At the onset of an outbreak, acute infectious diseases such as the zombie virus initially affect large, densely-packed cities most severely. For a zombie outbreak, this leads to mass exodus from the city, and the spread of the disease is increasing significantly. Next, the suburban areas and smaller cities are affected, and once again, citizens flee populated areas. Eventually, the distribution of population becomes more homogeneous. Small groups of susceptibles, such as those in *The Walking Dead*, survive by avoiding the infected population. Usually, these groups of humans also avoid contact with other groups of susceptibles as other groups are most likely desperate for resources and may become hostile. Clearly, the physical size of the contiguous United States will not change, but the population size will be in constant flux. Thus, as the population of infecteds and susceptibles changes in size, so does the density of this population. Therefore, a density-dependent transmission rate will be used for this model, and the transmission term is βSZ.

3.3 Model Development

For long-term modeling, birth rate and death rate should be introduced into the model. Let π be the birth rate, and let δ be the death rate. Many previous models have excluded birth rates and death rates, especially those modeling short periods of time, including the models of Witkowski [9] and Calderhead [2]. Including these rates will create a more robust model which can be applied to longer time periods. This can be important for modeling



S = number of individuals that are susceptible
Z = number of individuals that are zombies
R = number of individuals that are removed
π = birth rate
δ = death rate
β = bite rate
γ = zombie kill rate

Figure 2: The basic SZR flow diagram

immunization, especially when this method of survival does not ensure a quick end to the zombie apocalypse. In this model, birth rates and death rates will remain constant. The flow diagram of the SZR model is shown in Figure 2.

We now begin to build up the model from the basic form to its complete form. From here on, we will not include the R compartment in any models as it does not affect the calculations of future S and Z values. We refer to this model as the SZ model, shown in Figure 3. We see that the growth of human population is constant in Munz's model in Figure 4. Also, we see zombies

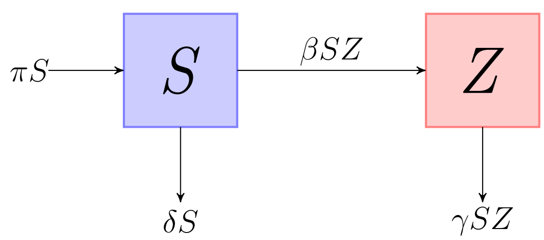


Figure 3: The SZ flow diagram

leaving the Z compartment via γSZ and some percentage of these zombies re-entering the Z compartment via λR. This reanimation is not consistent with any known zombie behavior in any movie, book, or television series. This dis-

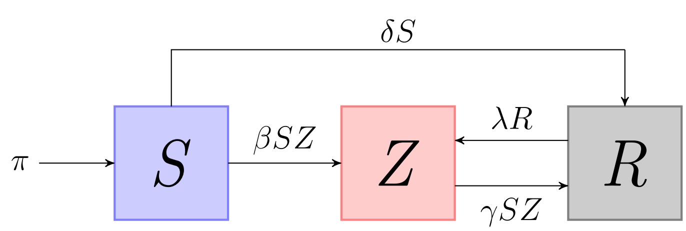


Figure 4: Flow diagram from Munz's paper [8]

crepancy has been discussed in modeling courses at many universities including Texas A&M, Bryant University, and Colorado State University. Also, these discussions agree that the number of susceptibles added to the model should not be constant, as seen in Figure 4, but the growth should be dependent on the number of susceptibles, as in Figure 3. These classroom discussions have increased interest in the area of mathematical modeling.

Since the zombie virus is also air-born, we move the δS arrow in between the S and Z showing that humans become zombies even after dying from non-zombie related causes as seen in Figure 5. Also, we see that some percentage

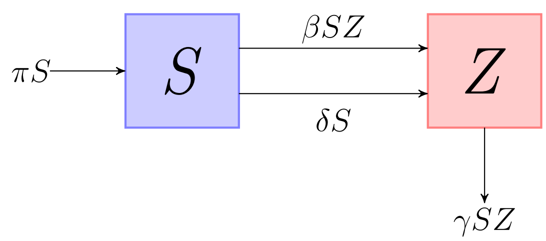


Figure 5: The SZ flow diagram with *The Walking Dead* airborne virus

of humans die and never become zombies. This can happen one of two ways. Since a zombie's natural instinct is to consume human flesh, many humans are eaten. If the brain is consumed, the individual will not be able to transform into a zombie. Secondly, many humans that have received a zombie bite are mercifully "pithed" (intentionally receive massive brain damage) by friends and family before turning into zombies. Not all susceptible are devoured or pithed, so a percentage, α, transit to the Z class as seen in Figure 6. The *pithing percentage* can be calculated as $1 - \alpha$.

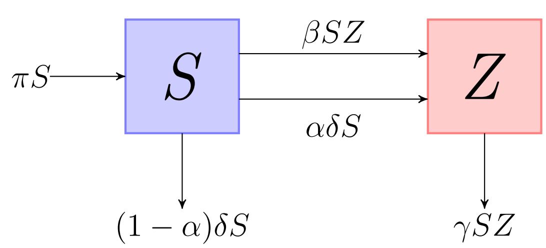


Figure 6: The complete SZ flow diagram with airborn virus and pithing

3.4 Parameter Values

Zombie outbreaks happen quickly. Usually, within a few days or less, the outbreak has either been quelled or become epidemic. Therefore, time will be expressed in hours. In zombie films, the plot is centered around these opening few days. This is usually the case for most zombie plots, but AMC's *The Walking Dead* excludes these opening hours and begins the storyline after the initial chaos is over and the epidemic has set in. To find a zombie bite rate, zombie films will need to be examined. To find birth and death rates in the contiguous United States, data from 2011 is used.

The United States Census Bureau lists the 2011 U.S. population at 311,587,816 [13]. Subtracting out the population of Hawaii and Alaska, the U.S. contiguous population is 309,485,827 [12]. The National Center for Health Statistics (NCHS) reports that 3,923,178 births occurred in the contiguous U.S. in 2011 [10]. Hence the hourly *birth rate* is

$$\pi = \frac{\dfrac{3,923,178 \text{ susceptibles}}{309,485,827 \text{ susceptibles}}}{365 \times 24 \text{ hours}} \approx 1.45 \times 10^{-6} \text{ per hour.}$$

The NCHS's preliminary report indicates that 2,499,401 deaths occured in the contiguous U.S. in 2011 [5]. Hence the hourly *death rate* is

$$\delta = \frac{\dfrac{2,499,401 \text{ susceptibles}}{309,485,827 \text{ susceptibles}}}{365 \times 24 \text{ hours}} \approx 9.2 \times 10^{-7} \text{ per hour.}$$

McCallum states that the transmission coefficient is the most difficult parameter to estimate [7]. The β parameter can be determined by two commonly used methods, either by experiment or by field observation. Fortunately, data has already been collected by Caitlyn Witkowski and Brian Blais in their paper titled *Bayesian Analysis of Epidemics - Zombies, Influenza, and Other Diseases* [9]. This paper analyzed two movies, *Shaun of the Dead* and *Night of the Living Dead*, approximating population estimates from the films at various time-points. The paper claims that zombies from *The Walking Dead* are most similar to those of *Shaun of the Dead*, so zombie counts taken from *Shaun of the Dead*, shown in Table 1, will be used. The paper also states that if we assume

Shaun of the Dead

t (in hours)	0.0	3.0	5.0	6.0	8.0	10.0	22.0	22.2	22.5	24.0
Z	0	1	2	2	3	3	4	6	2	3

t (in hours)	25.5	26.0	26.5	27.5	27.75	28.5	29.0	29.5	31.5
Z	5	12	15	25	37	25	65	80	100

Table 1: Zombie counts taken from *Shaun of the Dead*

that the zombies grow unrestricted and the population of the susceptibles is initially large enough to not be significantly affected, we can approximate the dynamics of zombie population growth as:

$$Z' = \beta S_0 Z$$

which has the solution

$$Z = Z_0 e^{\beta S_0 t}$$

or a slope of βS_0 on a $\log Z$ plot. We can transform the data in Table 1 by taking the log of Z values. Transformed data from *Shaun of the Dead* is shown in Table 2, and the plot is shown in Figure 7.

During the first ten hours of *Shaun of the Dead*, Shaun and his friends do not yet notice the zombie outbreak, and zombies appear infrequently during this span of time in the movie. From hour ten to hour twenty-two, no zombies

Shaun of the Dead

t (in hours)	0.0	3.0	5.0	6.0	8.0	10.0	22.0	22.2	22.5	24.0
$\ln Z$	n/a	0.0	0.7	0.7	1.1	1.1	1.4	1.8	0.7	1.1

t (in hours)	25.5	26.0	26.5	27.5	27.75	28.5	29.0	29.5	31.5
$\ln Z$	1.6	2.5	2.7	3.2	3.6	3.2	4.2	4.4	4.6

Table 2: Transformed data from *Shaun of the Dead*

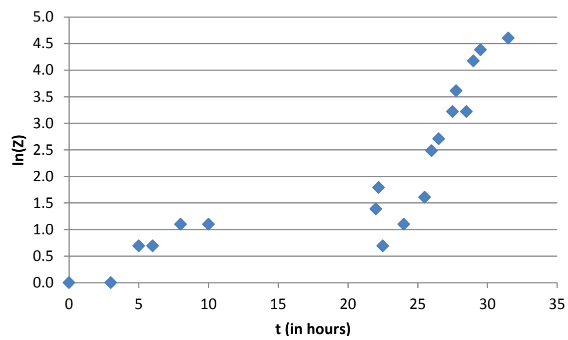


Figure 7: Plot of transformed data from *Shaun of the Dead*

are found on-screen, so a reliable count cannot be taken before hour twenty-two. Witkowski and Blais discard the first twenty-two hours of data, and the transformed data from the remaining hours in Table 2 is plotted in Figure 8. This plot shows linearization of the data, demonstrating exponential growth.

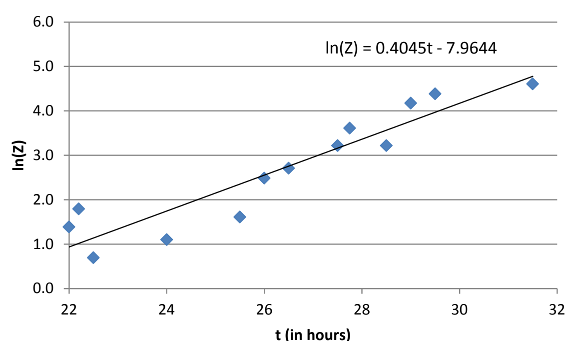


Figure 8: Plot of transformed data demonstrating exponential growth

Hence,

$$\ln Z = 0.4045t - 7.9644.$$

Exponentiating both sides,

$$Z = e^{0.4045t - 7.9644} = Z_0 e^{0.4045t}$$

where $Z_0 = e^{-7.9644} \approx 0$. This yields

$$\beta S_0 = 0.4045.$$

For the contiguous United States,

$$\beta = \frac{0.4045}{309{,}485{,}827} \approx 1.31 \times 10^{-9}.$$

Lastly, we discuss units of measurement for the bite rate β and zombie removal rate γ. Since the transmission term represents the number of susceptibles bitten in one hour, we have

$$\beta S_0 Z_0 = \frac{\#\text{ (of } S\text{) bitten}}{1\text{ hour}}.$$

Recall that $Z_0 = 1$ zombie. Hence,

$$\beta = \frac{\dfrac{\#\text{ (of } S\text{) bitten}}{S_0 Z_0}}{1\text{ hour}} = \frac{\dfrac{\%\text{ (of } S\text{) bitten}}{1\text{ zombie}}}{1\text{ hour}}.$$

Let S_p and Z_p be the number of susceptibles and zombies at the end of the doomsday scenario. In the doomsday scenario, $S_p = 1$ susceptible and $Z_p \geq 0$ zombies. Then

$$\gamma S_p Z_p = \frac{\#\text{ (of Z) removed}}{1\text{ hour}}$$

which yields

$$\gamma = \frac{\frac{\#\text{ (of } Z\text{) removed}}{S_p Z_p}}{1 \text{ hour}} = \frac{\frac{\%\text{ (of } Z\text{) removed}}{1 \text{ susceptible}}}{1 \text{ hour}}.$$

The pithing percetage will largely depend upon educating the public about the transmission of the solanum virus. If the public is aware of this option, relatives and loved ones will be motivated to join the pithing movement. The zombie removal rate, γ, will also be explored in later sections. In this paper, parameters are constant unless otherwise indicated.

3.5 Adding Latency

All humans must die before becoming a zombie. From *The Walking Dead*, we learn that in every case, there is a latent period ranging from 3 minutes to 8 hours [11]. To model this latent period, we introduce an additional compartment, E (Expired), into the model. This compartment represents the number of humans who are deceased and can turn into zombies. The *turning rate*, λ, is calculated in the same manner as the recovery rate in the SIR mode as the inverse of the average time an individual spends in the compartment. Assuming the average latent period is 4 hours, $\lambda = 0.25$ per hour. Recall $(1-\alpha)$ is the percentage of expired individuals that is pithed. The individuals in E that are not pithed, αE, either remain in E or transition to Z. The term $(1-\alpha)E$ has no units of time, so let $\chi = \%$ of E pithed that are removed from the model per hour. Since all pithed are removed from the model, $(1-\alpha)\chi E$ is the number of individuals in E pithed per hour. The flow diagram of the model with latency is shown in Figure 9. The flow diagram in Figure 9 gives

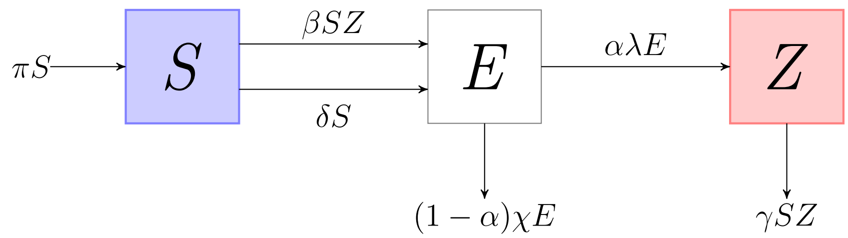


Figure 9: The complete SEZ model flow diagram

rise to the following system of differential equations:

$$\frac{dS}{dt} = \pi S - \beta SZ - \delta S \tag{1}$$
$$\frac{dE}{dt} = \beta SZ + \delta S - (1-\alpha)\chi E - \alpha\lambda E \tag{2}$$
$$\frac{dZ}{dt} = \alpha\lambda E - \gamma SZ \tag{3}$$

4 Analysis

4.1 Numerical Method

For each flow diagram, difference equations can be formulated from the differential equations. We now set up the numerical system for the SEZ model in Figure 9. From equations 1, 2, and 3, the SEZ model has the following system of difference equations:

$$S_{n+1} = S_n + \pi S_n - \beta S_n Z_n - \delta S_n$$
$$E_{n+1} = E_n + \beta SZ + \delta S - (1-\alpha)\chi E - \alpha\lambda E$$
$$Z_{n+1} = Z_n + \alpha\lambda E_n - \gamma S_n Z_n.$$

We can express the SEZ model as a dynamical system:

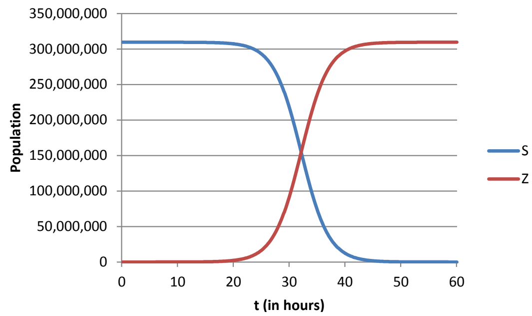
$$
\begin{aligned}
S_{n+1} &= S_n + \left[\pi(S_n + V_n) - \beta S_n Z_n - \delta S_n - \zeta S_n\right] \Delta t \\
E_{n+1} &= E_n + \left[\beta SZ + \delta S - (1-\alpha)\chi E - \alpha\lambda E\right] \Delta t \\
Z_{n+1} &= Z_n + \left[\alpha\lambda E_n - \gamma S_n Z_n\right] \Delta t,
\end{aligned}
$$

where $S_0 = 309,485,827$, $E_0 = 0$ and $Z_0 = 1$. The Runge-Kutta Method (RK4) will be used to compute numerical solutions to the model.

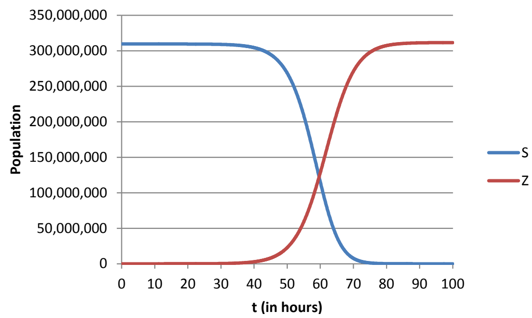
4.2 Latency and Pithing

Figure 10 compares the graphs of numerical solutions to the SZ and SEZ models in which we assume $\alpha = 1$ and $\gamma = 0$. The SEZ model is a more accurate picture of the zombie apocalypse and approximately doubles the time before annihilation of the human race. These extra hours are important for the human race to survive a zombie outbreak. Of course, since $\alpha = 1$ and $\gamma = 0$, this comparison between models assumes no response from the susceptibles. Even so, latency is clearly an important factor in determining how a zombie outbreak will unfold.

For most families, one unpleasant outcome of a zombie outbreak is watching loved ones become zombies. Fortunately, there is a remedy for the recently deceased. Ceremonial pithing, the act of intentionally causing severe trauma to the brain of the recently deceased, can be used to prevent family and friends from becoming zombies. It is important for this plan to be discussed with each group of family and friends. Of course, pithing is only suggested as a method of use after a human had expired. Since the deceased can begin to turn within three minutes, the ceremony must be performed with haste, although care must be taken to prevent accidental homicide.



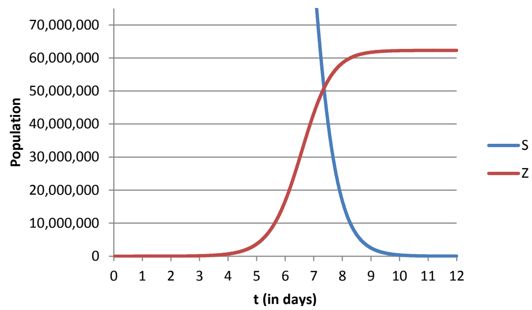
(a) SZ Model



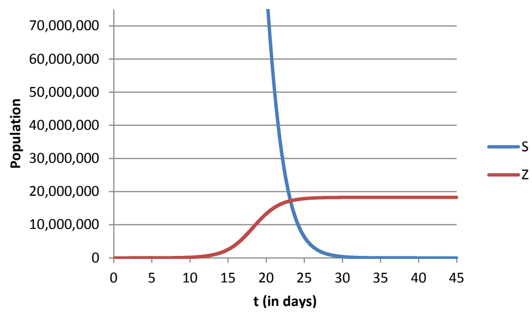
(b) SEZ Model: $\lambda = 0.25$

Figure 10: Comparison of SZ and SEZ models: $\alpha = 1.0$, $\gamma = 0$

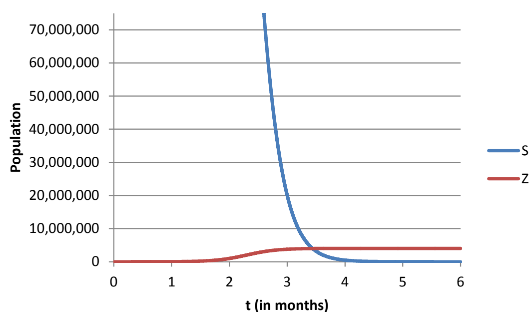
This plan can be implemented with great success, not only for small groups, but also for the larger population. In Figure 11, we see that an increased pithing percentage (recall that the pithing percentage is $1 - \alpha$) not only delays the drop in human population but decreases the final number of zombies. With a strong pithing plan in place, the human population still is annihilated, but at a much later time. More importantly, the zombie population is significantly decreased when the pithing percentage is increased. As with all communicable diseases, education is the best defense. Public announcements should include pithing procedures and should be made by all forms of broadcast.



(a) $1 - \alpha = 0.5$



(b) $1 - \alpha = 0.8$



(c) 1-$\alpha = 0.95$

Figure 11: SEZ models, various α, $\gamma = 0$

4.3 Equilibrium Analysis

We now attempt to find an equilibrium for Equations 1, 2, and 3. Setting Equation 1 equal to 0,

$$\frac{dS}{dt} = \pi S - \beta S Z - \delta S = 0$$

$$(\pi - \beta Z - \delta) S = 0.$$

Thus, susceptible equilibrium can only occur when $S = 0$ (the doomsday scenario) or when $\pi - \beta Z - \delta = 0$. Solving for Z,

$$\begin{aligned} Z &= \frac{\pi - \delta}{\beta} \qquad (4) \\ &= \frac{\dfrac{1.45 \times 10^{-6}}{1 \text{ hour}} - \dfrac{9.2 \times 10^{-7}}{1 \text{ hour}}}{\dfrac{\dfrac{1.31 \times 10^{-9}}{1 \text{ zombie}}}{1 \text{ hour}}} \\ &\approx 401.81 \text{ zombies.} \end{aligned}$$

Hence, when $Z > 401.81$ zombies, the susceptible population will decrease, and when $Z < 401.81$ zombies, the susceptible population will increase.

Setting Equation 2 equal to 0, substituting the equilibrium value for Z, and solving for E,

$$\frac{dE}{dt} = \beta S \left(\frac{\pi - \delta}{\beta} \right) + \delta S - (1 - \alpha)\chi E - \alpha \lambda E = 0$$

$$E = \frac{\pi S}{(1 - \alpha)\chi + \alpha \lambda}$$

Setting Equation 3 equal to 0 and substituting in the above value of E and the equilibrium value for Z,

$$\frac{dZ}{dt} = \alpha \lambda \left(\frac{\pi S}{(1 - \alpha)\chi + \alpha \lambda} \right) - \gamma S \left(\frac{\pi - \delta}{\beta} \right) = 0$$

$$\gamma = \frac{\alpha \beta \lambda \pi}{\big((1 - \alpha)\chi + \alpha \lambda\big)(\pi - \delta)}. \qquad (5)$$

When γ is set to this value, the system tends to an equilibrium in which

$Z = (\pi - \delta)/\beta$, and S is in proportion with E in accordance with Equation 2:

$$\beta S\left(\frac{\pi - \delta}{\beta}\right) + \delta S - (1-\alpha)\chi E - \alpha\lambda E = 0$$

$$\pi S = \left[(1-\alpha)\chi + \alpha\lambda\right] E. \tag{6}$$

When $\alpha = 0.2$, the zombie removal rate is $\gamma \approx 2.12 \times 10^{-10}$ per SZ per hour, and the system is at equilibrium by day 10. At this equilibrium, $S \approx 309,489,547$ individuals, $E \approx 5374$ individuals, and $Z \approx 402$ individuals. This equilibrium is plotted in Figure 12.

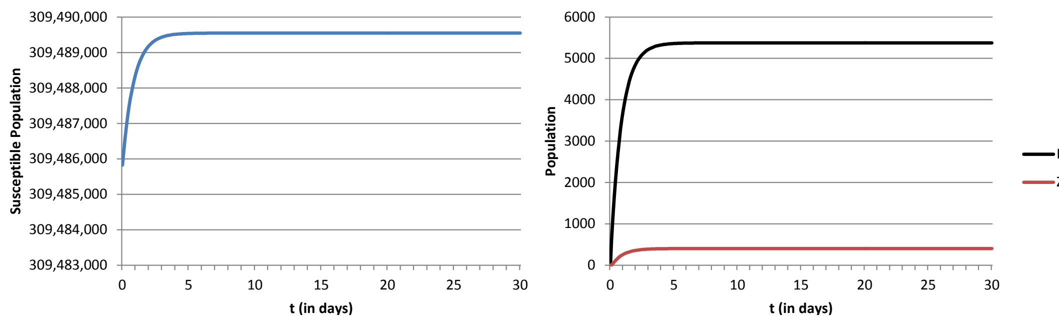


Figure 12: Equilibrium, $\alpha = 0.2$, $\gamma \approx 2.12 \times 10^{-10}$ per susptible per hour

A system that is stable with respect to that variable will then converge to equilibrium, but system that is asymptotically stable with respect to that variable will converge back to the original equilibrium values for each variable. To test the stability of equilibrium, we can change the value of each variable, S, E, and Z, when the system is at equilibrium and observe the results. We now change the values of the system in Figure 12 at day 15. Lowering the value of S to 309,000,000 at day 15, the system finds new equilibriums for S and E, but the Z equilibrium remains unchanged. These results can be seen in Figure 13. At this new equilibrium, $S \approx 308,999,995$, $E \approx 5366$, and $Z \approx 402$. Therefore, equilibrium is stable with respect to S.

Lowering the value of E at day 15 to 4000 individuals, we see that the

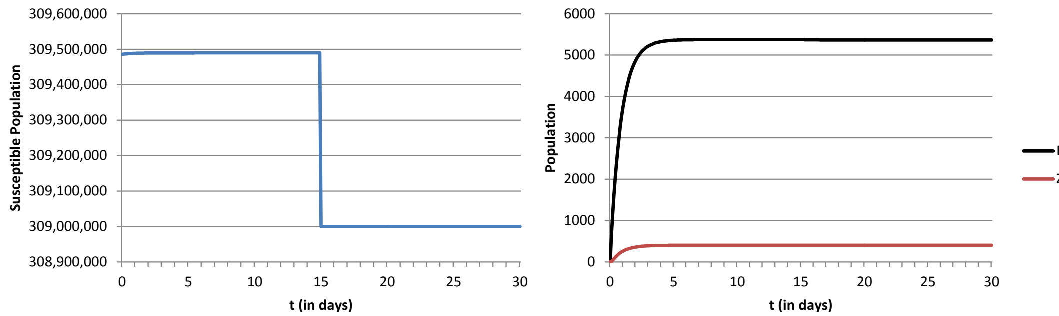


Figure 13: Stability test, changing the value of S

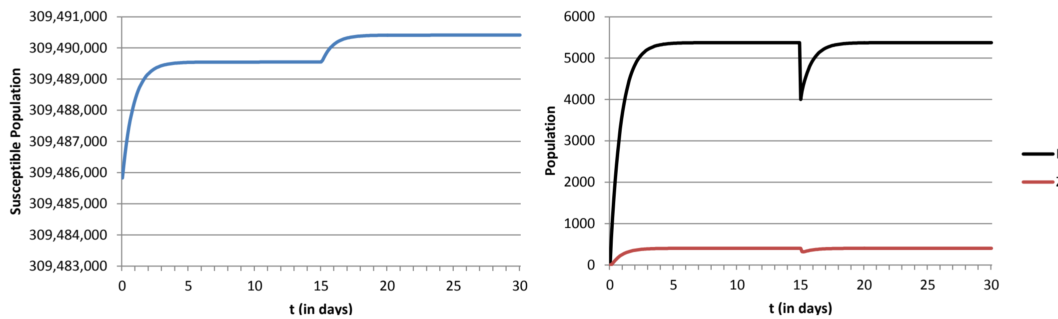


Figure 14: Stability test, change the value of E

system finds new equilibrium values for S and E. These results can be seen in Figure 14. At this new equilibrium, $S \approx 309,409,410$, $E \approx 5374.30$, and $Z \approx 402$. Therefore, equilibrium is stable with respect to E.

If we change the value of Z when the system is at equlibrium, we already know that the susuceptible population will increase or decrease accordingly. The value of γ has been calculated to ensure that the equilibrium value for Z is equal to $(\pi - \delta)/\beta$. Hence, S and E will find new equilibrium values when Z is changed, but Z will return to same equlibrium value. Therefore, the system at equilibrium is stable with respect to S and E and is asymptotically stable with respect to Z.

4.4 Zombie Removal

One of the best ways to control a zombie outbreak is to remove as many zombies from the system as possible. In conjunction with pithing, zombie removal can have encouraging results. Pithing lowers the rate at which susceptibles become zombies. Zombie removal lowers the occurrence of zombie bites (fewer zombies results in fewer bites). Knowing the appropriate zombie removal rate is a key to the survival of the human race. Equation 5 provides the minimum threshold for the γ parameter for survival of the human race, but this γ may not be attainable in the first weeks, months, or years of a zombie outbreak. Another minimum threshold can be found for γ that keeps the zombie population in check and delays the end of humanity.

We now find an equilibrium for the Z compartment only. In this scenario, the number of susceptibles will either increase or decrease, but a zombie kill rate, γ, can be determined to achieve a specific zombie population at equilibrium. Let Z^* denote the desired zombie equilibrium value. Setting Equation 3 to zero and solving for E,

$$\alpha\lambda E - \gamma S Z^* = 0$$

$$E = \frac{\gamma S Z^*}{\alpha\lambda}$$

In the equation above, E is in direct proportion to SZ^*. Setting Equation 2 to zero, substituting in the above value for E, and solving for γ,

$$\beta S Z^* + \delta S - (1-\alpha)\left(\frac{\gamma S Z^*}{\alpha\lambda}\right) - \alpha\lambda\left(\frac{\gamma S Z^*}{\alpha\lambda}\right) = 0$$

$$\gamma = \frac{(\beta Z^* + \delta)\alpha\lambda}{(1-\alpha+\alpha\lambda)Z^*} \tag{7}$$

This γ determines an equilibrium for the Z compartment in terms of the γ parameter. For any particular constant α, an appropriate γ can be calculated to achieve the target equilibrium for zombies, Z^*.

Equation 7 does not ensure survival of the human race. Zombie removal by an unorganized civilian population will result in a poorly executed response to the outbreak. At best, this civilian response may be able to hold the zombie population to a few thousand zombies, but in the long-run this effort is fruitless. Education and organization is the correct response, but both are usually in short supply during any apocalyptic event. Let $\alpha = 0.2$ and $Z^* = 20,000$, $\gamma \approx 7.96 \times 10^{-11}$ per susceptible per hour. Figure 15 shows the susceptible population decreasing by half after 4 years and approaching zero near the 30 year mark and the zombie population leveling off shortly after 4 months. Let $\alpha = 0.2$ and $Z^* = 3000$, $\gamma \approx 9.45 \times 10^{-11}$ per susceptible per hour. Figure 16 shows the susceptible population decreasing by half after 30 years and approaching zero near the 150 year mark and the zombie population leveling off sometime after the 20 days. A lower zombie equilibrium allows more time for susceptibles to organize and respond and keeps the zombie population below equilibrium for an extended period of time. If low zombie equilibrium is maintained, an early response will face fewer zombies with more susceptibles.

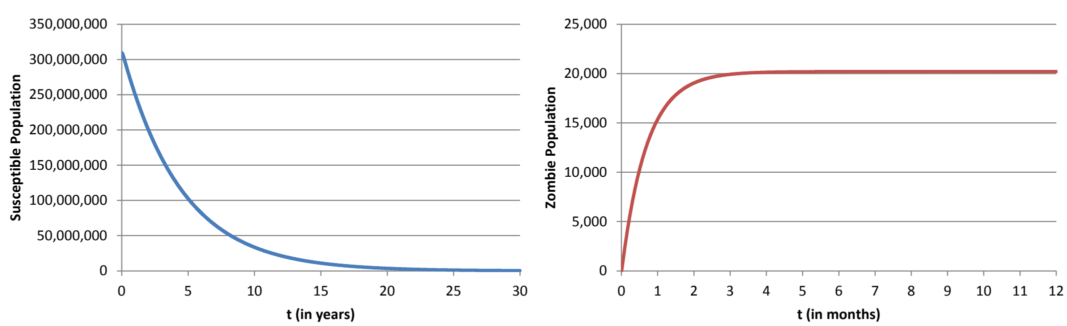


Figure 15: $Z^* = 20,000$, $\gamma \approx 7.96 \times 10^{-11}$ per susceptible per hour

From Equation 4, we know that the susceptible population will increase

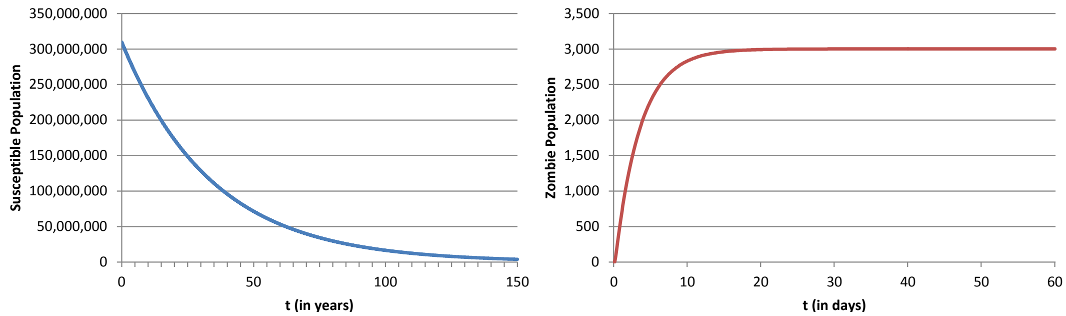


Figure 16: $Z^* = 3000$, $\gamma \approx 9.45 \times 10^{-11}$ per susceptible per hour

only if $Z^* < (\pi - \delta)\beta \approx 401.81$. If we let let $\alpha = 0.2$ and $Z^* = 401$, we calculate a zombie removal rate using Equation 7 that will save the human race:

$$\gamma = \frac{(\beta Z^* + \delta)\alpha\lambda}{(1 - \alpha + \alpha\lambda)Z^*} \approx 2.12 \times 10^{-10} \text{ per susceptible per hour.}$$

Note that this last γ value is approximate to the equilibrium γ value found in Figure 12. But if $\alpha = 0.5$ and $Z^* = 401$, then $\gamma = 7.21 \times 10^{-10}$ per susceptible per hour, and when $\alpha = 1.0$ and $Z^* = 401$, then $\gamma = 3.61 \times 10^{-9}$ per susceptible per hour, reinforcing the fact that a response by force should be used in conjunction with aggressive pithing practices. The graphs in Figure 17 show such a response using the rate calculated for $\alpha = 0.2$ beginning at the fiftieth hour.

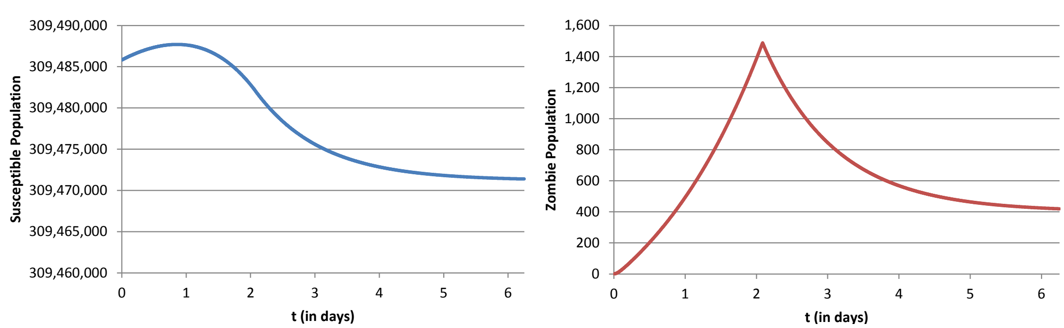


Figure 17: $\alpha = 0.2$, $Z^* = 401$ after 50^{th} hour

4.5 Zombie Decimation

Without an early, organized response, a sufficient zombie removal rate will not be established in time to thwart a doomsday event. The zombie removal rate in Figure 17 is significantly greater that those found in Figure 15 and Figure 16. The susceptible population may not be able to initially respond with such force, so a method will need to be devised by which force can be increased gradually and methodically. Many methods can be created, but the method of *decimation* will be presented in this paper.

Senior commanders of the Roman Army used decimation, or reduction by one-tenth, as a form of military discipline for deserters and other disobedient soldiers. Groups of ten soldiers were formed, and one of the ten was selected by lot and beaten to death by the other nine. More recently, the White troops of the Finnish Civil War used this method in 1918 to punish captured Red troops. Decimation can also be used during the initial period of the zombie outbreak to allow the military to ramp up war efforts gradually.

Suppose that $\alpha = 0.5$. If $\gamma = 0$ for the first 48 hours of the outbreak, a sufficient zombie removal rate to be implemented at hour 48 can be calculated using $0.9 \times Z_{48}$ as follows:

$$\gamma_1 = \frac{\big(\beta(0.9Z_{48}) + \delta\big)\alpha\lambda}{(1-\alpha+\alpha\lambda)(0.9Z_{48})} = 1.519 \times 10^{-9} \text{ per susceptible per hour}$$

where $Z_{48} = 20,000$ is the number of zombies at hour 48. This initial zombie removal rate can be used for a period of time, say from hour 48 to hour 60, but a new zombie removal rate will need to be calculated to prevent further decline of the susceptible population. The next zombie removal rate can be calculated by multiplying the first zombie decimation by 0.9. Hence, we use $0.9^2 \times Z_{48}$ in Equation 7. This second rate will be used from hour 60 to hour 72. Every 12 hours, a new zombie removal rate is implemented using the general formula

$$\gamma_n = \frac{\big(\beta(0.9^n Z_{48}) + \delta\big)\alpha\lambda}{(1 - \alpha + \alpha\lambda)(0.9^n Z_{48})}$$

The decimation method can be repeated until

$$Z^* < \frac{\pi - \delta}{\beta} \approx 401.81 \text{ zombies.}$$

At this point, $dS/dt \geq 0$, the zombie removal rate does not necessarily need to be reduced further to ensure survival and can be fixed at a comfortable number. Figure 18 shows graphs of the zombie and susceptible populations after intervention beginning at hour 48 using the decimation method.

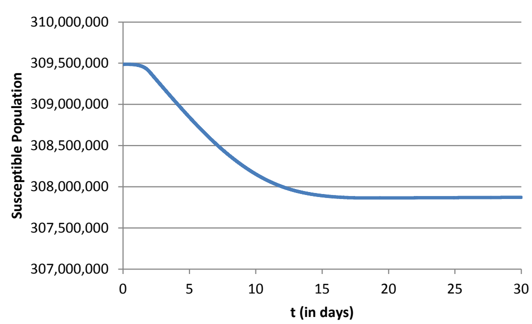
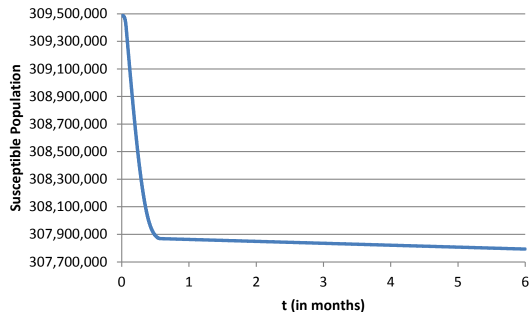
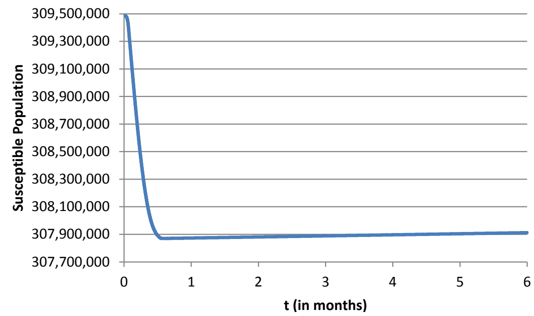


Figure 18: Decimation: $\alpha = 0.5$, $Z_{48} = 20,000$

As proved in Section 4.3, the susceptible populations in Figures 18 are not at equilibrium. In Figure 19, note that the susceptible population is increasing when $Z^* = 375$ and decreasing when $Z^* = 450$. Small changes in the value of Z^* create large changes in the value of S, a sign that the model is unstable with respect to Z^*. If the decimation process is stopped before the zombie population decreases below $Z^* \approx 401.81$, as in Figure 19a, the susceptible population will continue to decrease. If the decimation process is continued until $Z^* > 401.81$, as in Figure 19b, the human population will grow more quickly. Calculations will be made to allow for errors in data collection and computation, so the decimation method will run sufficiently below the Z^* threshold.



(a) $Z^* = 450$



(b) $Z^* = 375$

Figure 19: Small changes in Z^*

4.6 Vaccination

Pithing and decimation are powerful tools in the war against the solanum virus. These depend upon the military and the general public to control the outbreak. If the decimation quotas cannot be met, or if the general public cannot sufficiently refine pithing practices, science can provide another method to combat the zombie population.

Vaccination can be added to the model with an additional compartment, but several assumptions must be made. First, we consider how the vaccination will work. Will the children of vaccinated mothers need to be vaccinated? Will all vaccinated humans never become zombies under any circumstances? Will the vaccinated die from a zombie bite? We shall assume that all children will be born unvaccinated. Once vaccinated, a human will never become a zombie, but a zombie bite is still fatal.

Vaccination also requires many logistical assumptions. First, we assume that a vaccination can be found quickly. Generally, vaccinations are not easily developed, and research facilities must be secured from zombification. Second, production of the vaccine will take time and resources, both of which are in short supply during the onset of the outbreak. Finally, distribution will be a difficult task. We assume that the Center for Disease Control has prepared for such an outbreak, and these difficulties are minimal.

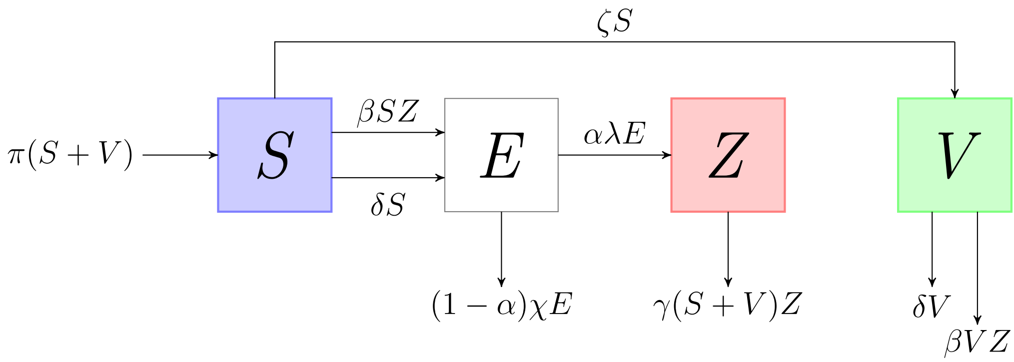


Figure 20: The $SEZV$ model flow diagram

Let ζ be the percentage of the susceptible population vaccinated each hour. Also, the number of non-bite related deaths in V is δV, and the number of deaths by zombie bite is βVZ. Note that the birth rate and zombie removal rate are now dependent on S and V, as seen in Figure 20. It is unlikely that a vaccination will be available within the first few hours, days or even weeks of the zombie outbreak. In fact, there is a strong possibility that a vaccination may never be found. Suppose that a scenario, such as the one in Figure 15, has

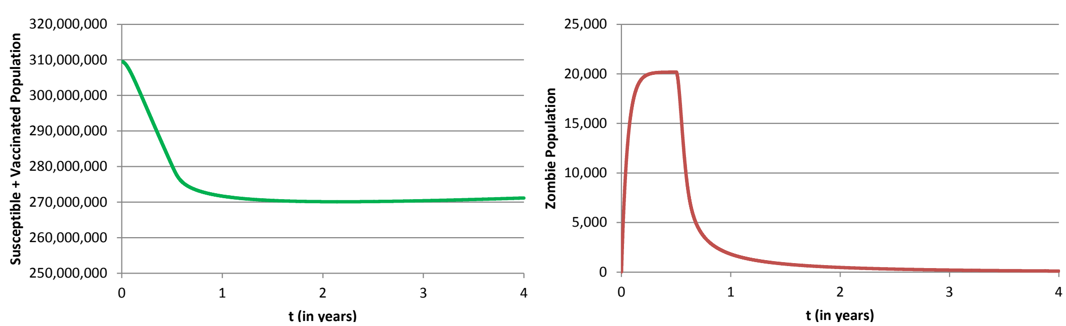


Figure 21: Vaccination beginning at month six

potentially doomed the human race. If a vaccination can be created relatively quickly, the human race might not be lost. In this scenario, let $\zeta = 0.00006944$ of S per hour, or about 5% per month. If vaccination is administered on the first day of the sixth month, and a continuous supply of vaccine is provided, then the human population is saved. These results are shown in Figure 21. If the vaccination program begins at year two, the total population of the human race is lowered further but still is able to recover as seen in Figure 22. Most importantly, vaccination decreases the value of S, and therefore decreases the rate at which the zombie population can increase.

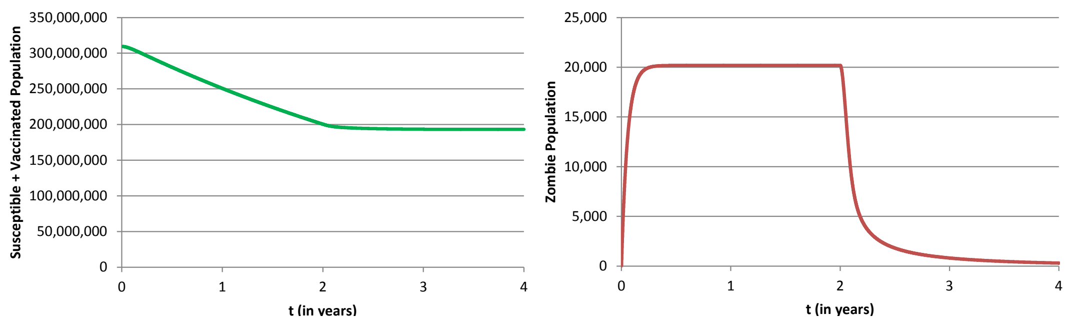


Figure 22: Vaccination beginning at year two

5 Conclusion

The zombie apocalypse may occur at any time without warning, and preparation is the key for survival of the human race. Pithing is not a method by which the human race can survive an outbreak, but pithing used in conjunction with an organized zombie removal response can have very positive results.

The zombie population will need to be kept at 401 zombies or less to prevent the susceptible population from decreasing. This can be accomplished by calculating an appropriate zombie removal rate. Recalculating the zombie removal rate, γ, periodically is practical method to gradually scale up military efforts. With an adequately high pithing percentage, the zombie removal rate can be set at an economically and socially comfortable level.

Vaccination has been shown to stop a zombie outbreak, but only if a vaccine can be developed and distributed in time. Again, preparation is key to a successful vaccination program. Without such preparation, a vaccination may not be found in a timely manner. Likewise, without pithing practices and zombie removal, a vaccination will most likely not be found in time to save the human race.

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[13] United States Census Bureau, Population Division. *Table 1. Monthly Population Estimates for the United States: April 1, 2010 to October 1, 2013 (NST-EST2012-01)*, 2012.