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The Four Color Theorem: A Possible New Approach

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The Four Color Theorem: A Possible New Approach

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Abstract

The goal of this thesis is to explore the topic of graph coloring and expand on existing ideas in the field of Graph Theory. These developments will then be used to provide a possible approach in proving the 4 – color theorem that was made famous by Guthrie in the 1800's.

Since the theorem was presented, many proofs were presented and eventually disregarded for one reason or another. Today, the types of proofs that are considered correct all rely on a computer. The first of this kind was set forth by Appel and Haken in 1977. [4] The driving idea behind their proof was exhaustive analysis. A different approach will be taken here.

The 4 – color theorem stated is: "Any finite, planar graph can be colored using 4 (at most) colors in such a manner that no adjacent vertices will share the same color." While a complete proof of the theorem may not be possible to complete in this thesis, an intuitive idea will be presented that has potential to be expanded on in the future.

Review of Literature

The 4 – Color Theorem was first made popular in the 1800's. It was presented as a statement in regards to map coloring. It was questioned whether all maps drawn in the plane can be colored with 4 colors and in a manner in which all adjacent countries are a different color. This was first knowingly questioned by Francis Guthrie who was a student of Augustus DeMorgan. He discussed it with DeMorgan and after they could not come to a conclusion, it was brought to Sir William Rowan Hamilton. [3]

The first claimed proof came from Alfred Kempe in 1879. [1] That proof did not last long, however. In 1890 P.J. Heawood found an error in Kempe's proof. Although, he did find that the mistaken proof had solid mathematics in it and used this idea to prove what is known today as "The 5 – Color Theorem." [7]

It is not until recently that a somewhat acceptable proof has been presented. In 1977 Appel and Haken presented a lengthy proof. A summary of their proof is as follows:

> The proof sets out first to show that every plane triangulation must contain at least one of 1482 certain 'unavoidable configurations'. In a second step, a computer is used to show that each of those configurations is 'reducible', ie., that any plane triangulation containing such a configuration can be 4 – coloured by piecing together 4-colouringsof smaller plane triangulations. Taken together, these two steps amount to an inductive proof that all plane triangulations, and hence all planar graphs, can be 4-coloured. [2]

This proof set forth by Appel and Haken has received much criticism for its approach from the mathematical community. Their main criticism stemmed from the proof's use of a computer. Using a computer to prove a theorem was not an idea widely accepted at the time. These criticisms have been answered by Appel and Haken with another very lengthy paper that corrected errors and added more configurations. [5]

Since Appel and Haken's proof, a much shorter proof has been provided. However, this proof still relies on the same computer methodology. This proof was presented in 1997 by N. Robertson, D. Sanders, P.D. Seymour & R. Thomas. [6]

When asked, R. Thomas said this about the Appel and Haken's proof and the proof that he had a hand in.

> For the purposes of this survey, let me telescope the difficulties with the A&H proof into two points (1) part of the proof uses a computer and cannot be verified by hand, and (2) even the part that is supposedly hand-checkable has not, as far as I know, been independently verified in its entirety. … Neil Robertson, Daniel P Sanders, Paul Seymour, and I tried to verify the Appel-Haken proof, but soon gave up and decided that it would be more profitable to work out our own proof. …We were not able to eliminate reason (1), but we managed to make progress toward (2). [8]

This brings us to present day. No other popular or accepted proof has been written. Also, there has been no accepted proof of the 4 – Color Theorem written that does not utilize a computer. This provides a large opportunity for a working idea.

Overview of Approach

The approach that is taken with this has paper has many steps and stages that build lemmas and culminates with a partial proof.

The process begins with taking a planar graph and making it larger, which seems odd at first. The motivation behind this expansion was to guarantee that every vertex in the original graph was also a vertex in a K_4 subgraph. When a vertex is in a K_4 subgraph it is a vertex that has a "forced coloring." (Forced coloring will be properly defined.) This was done to help in the proofs of other crucial lemmas as well as in the deconstruction/construction process.

After the expansion on the original graph, the graph is taken apart. This is performed with an operation that is defined. The expanded graph is separated into distinct K_4 subgraphs. Next, the graph is put back together and colored at the same time using an (strong) inductive proof.

We know that a K_4 graph is 4 – colorable. This process attempts to show that construction does not matter in which manner we attach other K_4 subgraphs. The construction will always result in a 4 – colorable subgraph. When we combine this fact with all the other pieces that are stepped through, it is easy to see why (certain) planar graphs are 4 – colorable. However, there is one exception case where the construction method cannot completely prove the theorem. This results in a partial proof rather than a complete proof of the 4 – Color Theorem.

The goal is to continue working on this until the method useable for all 4 – colorable graphs.

Introduction of Operations

Throughout multiple steps in this paper, two new operations will be defined – vertex pasting and vertex separating. It will serve best to begin with a discussion on these operations and examples of their application.

Vertex Pasting ($\#^+$)

Given two graphs G and H, let $u \in V(G)$ and $v \in V(H)$. Vertex pasting pastes u and v together so it becomes a single vertex denoted (u, v). This implies that all adjacent vertices of u and v will be adjacent to (u, v). The operation on vertices is denoted $u\#^+v = (u, v)$. See an example in *Figure 1*.

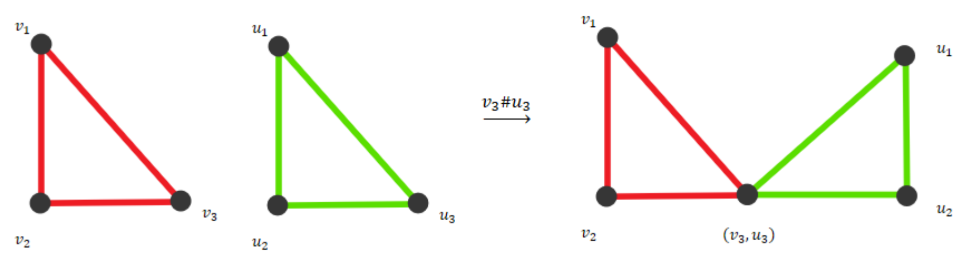

Figure 1

Vertex Separating ($\#^-$)

Let G be a graph and $u \in V(G)$. Vertex Separating must be applied to a cut vertex or cut set. Applying Vertex Separating to a cut vertex u will create i disconnected subgraphs G_i $(i = 1,2,3, \dots n)$ where $u \in V(G_i)$. That is, there will be n disconnected graphs (or components) of G. The operation on a cut vertex, u, will be denoted as $\#^-(u) \implies u_1 \in V(G_1), u_2 \in V(G_2) \dots u_n \in \mathrm{V}(G_n)$. This shows that by separating G at vertex u, G will be separated into n disjoint graphs where $u \in V(G_i), \forall i = 1, \dots, n$.

If Vertex Separating is applied to a cut set $\{u_1, \ldots u_m\}$, then it will produce i disconnected graphs where $\{u_1, \ldots u_m\} \in V(G_i)$ for $i = 1,2,3, \ldots, n$. The operation for cut sets is denoted $\#^-(\{u_1, \ldots u_m\})$.

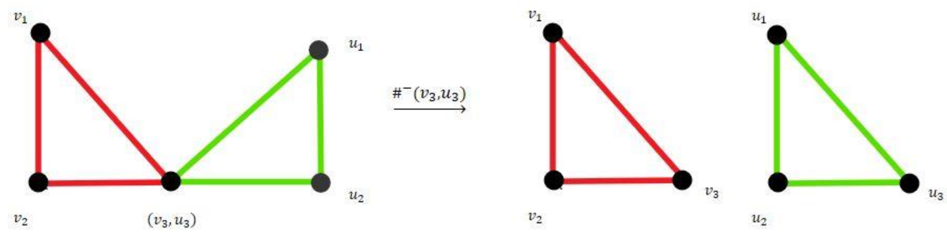

Figure 2

When *Vertex Separating* is applied to a vertex (or set of vertices) it will result in corresponding vertices. In *Figure 2*, if *Vertex Separating* is applied to vertex (v_3, u_3) it will separate the graph and vertex (v_3, u_3) will be present in the two separated graphs under the new names v_3 and u_3. Vertices v_3 and u_3 are considered ***Corresponding Vertices***. Every vertex that undergoes the operation of *Vertex Separating* will have a set of *Corresponding Vertices.* In other words, *Corresponding Vertices* is the set of new vertices that are derived from *Vertex Separating* one vertex.

Now that the operations are defined, the process of expanding the graph will be discussed. We will introduce a new term called ***Planar Graph K_4 Expansion (PGKE)***. This will be defined as a process that takes an already existing planar graph and enlarges both the vertex set and edge set by creating new K_4 subgraphs that are connected to the original graph. Recall that, a graph is said to be ***planar*** if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all.

We will assume that the graph G in which PGKE is applied will not have isolated vertices, for those are easily colorable.

Planar Graph K_4 Expansion

Let G be a planar graph with $|E(G)| = n$ where $n \geq 1$. Choose an arbitrary edge to begin with. This edge, $e_{i,1}$, will have two endpoints, $v_{i,1}$ and $v_{i,2}$. We will create two new vertices, $v_{i,3}$ and $v_{i,4}$ where $\{v_{i,3}, v_{i,4}\} \notin V(G)$. That is, we are adding two new vertices that were previously not in the graph. After we add the vertices we will have the vertex set $R_i = \{v_{i,1}, v_{i,2,} v_{i,3}, v_{i,4}\}$. Recall, only $v_{i,1}$ and $v_{i,2}$ are connected and they are connected by edge $v_{i,1}v_{i,2}$. We will now add five more edges, $S_i = \{v_{i,1}v_{i,2}, v_{i,1}v_{i,3}, v_{i,1}v_{i,4}, v_{i,2}v_{i,3}, v_{i,2}v_{i,4}, v_{i,3}v_{i,4}\}$. It is then realized that the subgraph $R_i \cup S_i = K_4^i$.

We should repeat this process for the remaining $n-1$ edges. After this is complete, we will have a new graph G' that contains at least n, K_4 subgraphs.

When adding the new vertices and edges, they should added in a manner that maintains the planarity of G. This is done so that after the new K_4 graph is added, the entire graph will remain planar.

In *Figure 3*, we see the **Planar Graph K_4 Expansion** process illustrated on the edge $v_{i,1}, v_{i,2}$ as outlined in the process above. On the left, we have the graph G. On the right we add a K_4 subgraph onto the edge $v_{i,1}, v_{i,2}$.

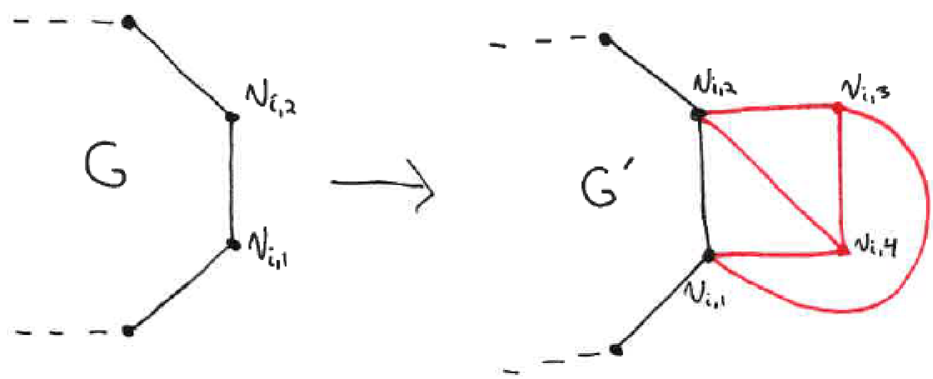

Figure 3

Also, demonstrated in *Figure 4* is what the **Planar Graph K_4 Expansion** would look like on the cycle K_3. Note, K_3 has three edges and therefore after the expansion is complete, it has (at least) three distinct K_4 subgraphs.

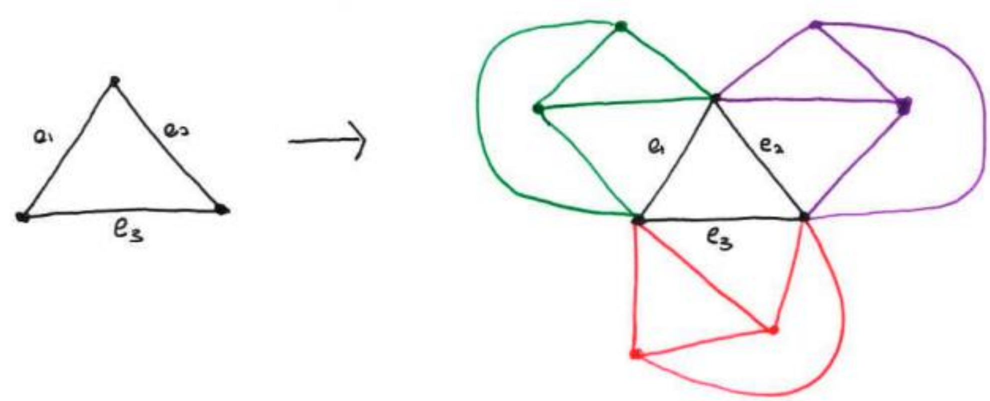

Figure 4

Since the graph has been expanded using **Planar Graph K_4 Expansion,** it provides the graph with certain properties. Now, new vocabulary will be introduced and certain observations will be drawn from the previous pages.

Free Vertex – Let G be a graph that is 4 – colored and let $v_i \in V(G)$. v_i is considered a free vertex if the color of v_i can be changed to at least one other color (from the 4 color set) and remain a proper 4 – coloring without changing the color of other vertices in G.

Forced Coloring- Given a planar graph G with $v_i \in V(G), v_i$ has a forced coloring if throughout all of the possible 4-colorings of G, v_i will never be a free vertex. In other words, changing the color of v_i will always force an adjacent vertex to change colors as well. This is important because it implies that it must be connected to at least three different vertices with different colors for all colorings. Those three vertices will all be different colors from each other as well.

4 – Mandatory (Sub)graph – A subgraph that is comprised entirely of vertices that have forced colorings. A mandatory (sub)graph must have at least 4 vertices. In a mandatory subgraph, every vertex has a forced coloring. Now, if we inspect the vertices of a graph that went through **Planar Graph K_4 Expansion**, we notice that every vertex is included in at least one K_4 subgraph and hence, every vertex has a forced coloring. This is due to the fact that any K_4 subgraph automatically exhausts all four colors on the four vertices in the K_4 subgraph due to its connectivity.

Since every vertex in a **Planar Graph K_4 Expanded** graph has a **forced coloring**, that would make every **Planar Graph K_4 Expanded** graph a **mandatory graph** as well. If we let G' be the name of the graph that is **Planar Graph K_4 Expanded** on the planar graph G, then we can draw the following conclusions about G'.

1) $G' = K_4^1 \cup K_4^2 \cup \cdots \cup K_4^n$. That is, G' is a composite of different complete graphs on four vertices. This is by design of G'. When G' was constructed, each of the individual n edges in the original graph G was given its own K_4 subgraph. Hence, every edge in G' is part of a K_4 subgraph and there will be (at least) n distinct K_4 subgraphs since G has n edges.
2) $V(G') = V(K_4^1) \cup V(K_4^2) \cup \cdots \cup V(K_4^n)$. This implies that the vertex set of G' will contain the same vertices as the vertex sets of the n, K_4 subgraphs. This follows from #1 above.
3) For all $v_i \in V(G')$, $v_i \in V(K_4^j)$. That is, every vertex in G' is also in a K_4 subgraph ($K_4 \subset G'$) for at least one j. This follows from #2 above.
4) For all of the K_4 subgraphs that are outlined in #1 above, only one edge from the original graph G will be in each of the K_4 subgraphs. This is also by design. In the expansion process, every edge that exists receives its own K_4 subgraph. If we observe the newly created K_4 subgraphs, only one original edge from G will exist in each of the n distinct K_4 subgraphs.

In the above paragraph, the special property of forced vertices is discussed. We can add onto this property and define what is called, *dependent vertices.*

Dependent Vertices

Dependent Vertices – Given a graph G with $|V(G)| \geq 5, v_i$ and v_j are dependent vertices if $col(v_i) = col(v_j)$ for all possible 4 – colorings of G.

A Note on Notation

When dealing with dependent sets of vertices, a special notation will be used. If vertex v_i and v_j are dependent, we will denote that by writing $dep(v_i, v_j)$. If v_k is also dependent on at least one of these vertices, then we could denote that as $dep(v_i, v_j, v_k)$. It a situation where more than two vertices are dependent, the transitive property does apply. For example, in the case where $dep(v_i, v_j, v_k)$, if the color of v_i controls the color of v_j and the color of v_j controls the color of v_k, then when v_i changes colors it will also force v_k to change colors.

Now, observations about dependent vertices through a series of lemmas and corollaries will be made. These observations will reveal more about the nature of dependent vertices and where they can exist. The goal is to show that they can only exist in a unique situation in which we can predict and thus making it possible to draw conclusions about the graph based only on the fact that they exist.

Dependent Subgraph Lemma

Let $dep(v_i, v_j)$ be a dependent set in the planar K_4 expanded graph G, then there exists two subgraphs $A = K_4$ and $B = K_4$ $(A \neq B)$ with $v_i \in A$ and $v_j \in B$. These can be seen in *Figure 5 & Figure 6.*

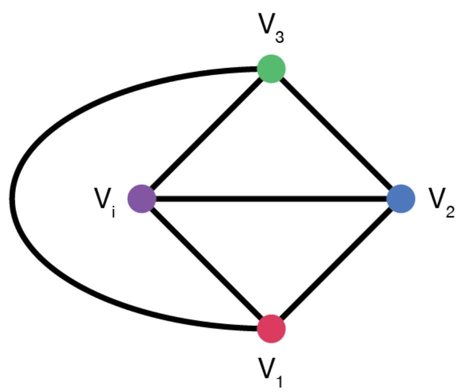

Figure 5

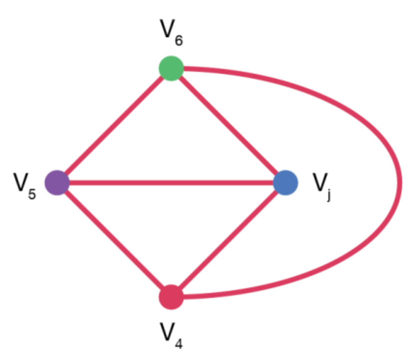

Figure 6

Proof: We know that v_i and v_j must be in a K_4 subgraph since they are vertices in a K_4expanded graph. Since, by definition of being dependent, $col(v_i) = col(v_j)$ then obviously v_i and v_j are not adjacent and therefore not in the same K_4 subgraph. ■

Next, the corollary that is stated will be left without complete proof. This is the "sticking point" of the paper. It appears that if this next corollary can be fully proven, the popular 4 – Color Theorem can be completely proved with the method this paper outlines. That will be revealed further into the paper.

Corollary 1.1 (to Dependent Subgraph Lemma):

Let $dep(v_i, v_j)$ be a dependent set in the planar K_4 expanded graph G, with two subgraphs $A = K_4$ and $B = K_4$ $(A \neq B)$ with $v_i \in A$ and $v_j \in B$.

If $V(A) \cap V(B) \neq \emptyset$ then $V(A) \cap V(B) = [V(A) \cup V(B)] \setminus \{v_i, v_j\}$, i.e. the intersection of A and B is filled with the three vertices from the subgraphs A and B that are not part of the dependent set. In this case, $|V(A) \cap V(B)| = 3$.

Note: The depiction in *Figure 7* is not intended to be a multi-graph. Also, the double edges should be considered as a single edge. It was intentionally illustrated in this manner to help the reader see how the two graphs were connected.

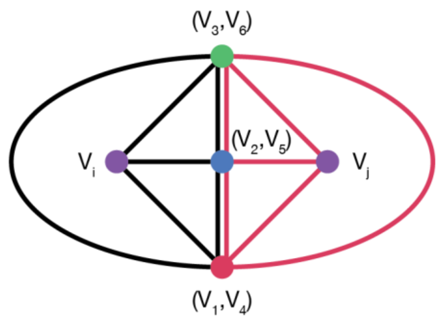

Figure 7

Proof (by cases):

We will prove this by illustrating the different cases: $|V(A) \cap V(B)| = 2$, $|V(A) \cap V(B)| = 1$. These cases will contradict the dependency of $dep(v_i, v_j)$.

Assume for contradiction that $V(A) \cap V(B) = [V(A) \cup V(B)] \setminus \{v_i, v_j, v_1, v_4\}$ that is to say that there are only two vertices in the intersection of graphs A and B, as shown in *Figure 8.*

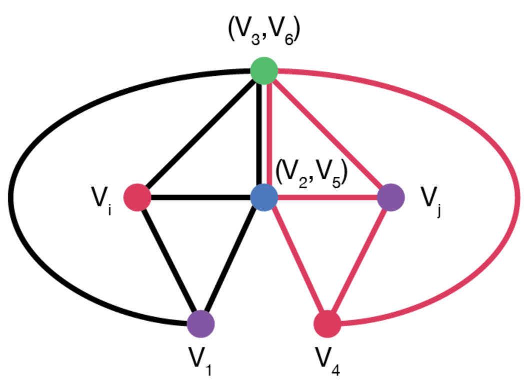

Figure 8

Obviously, from the representation above, both v_i and v_j are vertices in A and B respectively, but $col(v_i) \neq col(v_j)$ which is a contradiction of the definition of being dependent. The same argument works for:

$$V(A) \cap V(B) = [V(A) \cup V(B)] \backslash \{v_i, v_{j,} v_1, v_2, v_4, v_5\} = \{(v_3, v_6)\}$$

Thus, our only option is $V(A) \cap V(B) = V(A) \cup V(B) \backslash \{v_i, v_j\}$.

Remark: The case where $V(A) \cap V(B) = \emptyset$ is left without proof at this point. The corollary intentionally leaves this condition out. What is proven here is a weaker statement than what is necessary to prove the 4 - Color Theorem with this method.

Before moving on to corollary 1.2, two new terms that help build upon the properties of dependent sets will be introduced. These terms and corollary 1.2 will help determine the location of dependent sets just by knowing their existence.

Exterior – A vertex is considered to be on the exterior of the graph if a loop can be created at that vertex that encompasses the entire graph, while not crossing any other vertex or edge.

Interior – A vertex is on the interior of the graph if it is not on the exterior.

Corollary 1.2 (to Dependent Subgraph Lemma):

Let $dep(v_i, v_j)$ be a dependent set in the planar graph G. Let A and B as in the previous corollary. If $V(A) \cap V(B) \neq \emptyset$ then at least one vertex from the set $dep(v_i, v_j)$ is on the interior of G. Moreover, it is impossible to add an edge to G so that v_i and v_j are adjacent, if G is to remain planar.

Proof:

We know from the *Dependent Subgraph Lemma* that there exists two subgraphs $A = K_4$ and $B = K_4$ $(A \neq B)$ with $v_i \in A$ and $v_j \in B$. We also know that from *Corollary 1.1* that $V(A) \cap V(B) = V(A) \cup V(B) \backslash \{v_i, v_j\}$. This type of graph only has two types of representations, as seen in *Figure 9 & Figure 10*.

Representation 1:

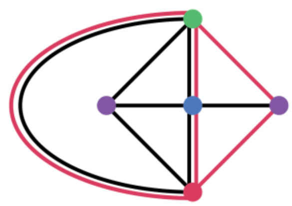

Figure 9

Representation 2:

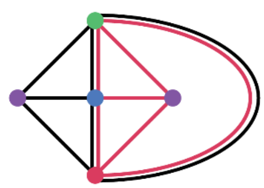

Figure 10

It is clear that from both representations that at least one vertex from the set $dep(v_i, v_j)$ is on the interior. It is also important to note that, since this is the case, it is impossible for v_i and v_j to be connected by an edge in such a way that the resulting graph is planar. ■

The Heavy Lifting

At this point, all of the key components of the graph that is being built has been established. Now, there will be a discussion on how to color this particular graph.

In the next proof, strong induction is used by removing a K_4 subgraph from the larger graph G'. We will see notation similar to this: $G' - K_4^i$. This notation indicates a graph that is a subgraph of G'. It is the graph G' with the edges from the K_4 subgraph removed from the graph G'. Also, the vertices that were added in PGKE to create subgraph K_4^i will also be removed.

Here is an example of this notation below in *Figure 11*.

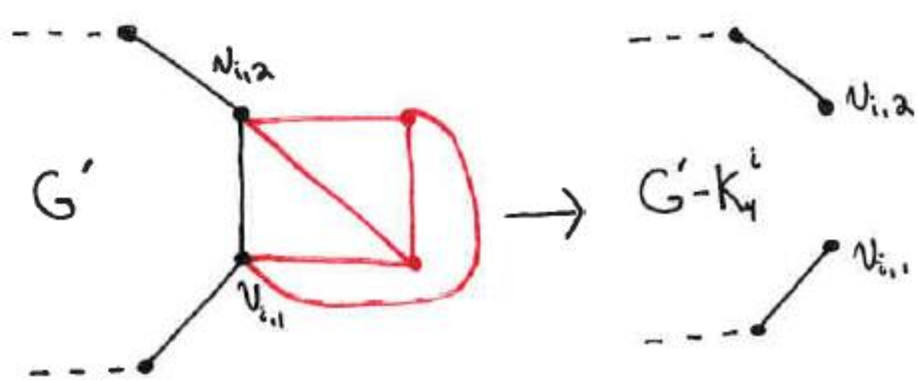

Figure 11

Planar Graph K_4 Expansion Coloring Lemma

Let G be a planar graph and let G' be the resulting graph of applying ***Planar Graph K_4 Expansion*** (PGKE) to graph G, that is $G \xrightarrow{PGKE.} G'$. It is then true that G' can be properly 4 – colored.

<u>Proof (by strong induction):</u> We know from ***observation 1*** that $G' = K_4^1 \cup K_4^2 \cup \cdots \cup K_4^n$ and this set of n K_4 subgraphs is distinct. We will induct on k, the number of K_4 subgraphs.

Base case:

$k = 1$. This case is obvious. It is a single K_4 graph which we know is 4 – colorable.

Inductive step:

We will let the hypothesis be true for all values of k such that $k < n$. Now, consider G', a graph with n K_4 subgraphs. Use *vertex separating* to remove an arbitrary K_4^i subgraph such that $K_4^i \in \{K_4^1, K_4^2, \dots, K_4^n\}$. This may result in the graph $G' - K_4^i$ being disconnected. However, the disconnected subgraphs will have fewer than n K_4 subgraphs and thus the hypothesis holds for each connected component of the disconnected subgraph. Also note, each component of $G' - K_4^i$ will be a collection of K_4 subgraphs.

Now, we will use *vertex pasting* to paste K_4^i onto its corresponding vertices in the graph $G' - K_4^i$ in order to arrive back at our starting graph G'. This creates two cases. Since the K_4^i that is removed is a K_4 subgraph that is added in the PGKE process, it will automatically have two corresponding vertices. The two cases that will be handled here deal with the coloring of the corresponding vertices.

Case 1: $G' - K_4^i$ and K_4^i have 2 corresponding vertices and they are not the same color in graph $G' - K_4^i$. Use *vertex pasting* to paste K_4^i with graph $G' - K_4^i$ at the 2

corresponding vertices. Since these vertices are different colors it does not matter that they are adjacent in graph K_4^i. Since the intersection of $G' - K_4^i$ and K_4^i in the graph G' only contains the two corresponding vertices, the remainder of graph K_4^i can be colored using the remaining 2 colors. Thus G' is 4 – colorable.

Case 2 (Can only be proved if corollary 1.1 and corollary 1.2 worked for all cases):

$G' - K_4^i$ and K_4^i have 2 corresponding vertices that are the same color in graph $G' - K_4^i$. Since the two corresponding vertices are the same color, they must not be adjacent.

Even though Case 2 assumes they are the same color, we know that a proper 4 – coloring of $G' - K_4^i$ exists where they are different. For if a coloring did not exist where the corresponding vertices' colors are not different, that would imply that the corresponding vertices are also dependent vertices, by definition.

However, these are not dependent vertices since after $G' - K_4^i$ and K_4^i are combined using graph pasting, the corresponding vertices are adjacent. *Corollary 1.2* states that, if vertices are dependent, it is impossible to add an edge to the graph that also makes the dependent vertices adjacent. Since after the graph pasting, the corresponding vertices will be adjacent, they clearly are not dependent. Thus, a coloring of $G' - K_4^i$ exists where the corresponding vertices are different colors. This refers back to *Case 1*.

Case 2 relies on *Corollary 1.2* being completely proven. This is required since the approach to proving *case 2* above relies on making the claim that dependent vertices cannot be adjacent. If fully proven, *Corollary 1.2* would show this. ■

It is clear that the above lemma cannot be proved for all cases. However, what we end up with is still useable but on a smaller scale. We proved that when you apply PGKE to a planar graph, it will be 4 – colorable if when adding back in the arbitrary K_4 graph, the ***corresponding vertices*** are not the same color. Since the goal was to prove the 4 – Color Theorem with all of these tools, this obviously means that the statement of the 4 – Color Theorem will have to be altered in order to fit what actually has been proven.

The Punch Line

The 4 – Color Theorem has been mentioned numerous time throughout this paper. The actual statement of the theorem is as follows: Every Planar graph G is 4-colorable. Unfortunately, the results of this paper are not strong enough to prove the original statement without modification. The statement will have to be modified to fit the results that this paper has achieved

(Modified) 4 – Color Theorem:

Every Planar graph G is 4-colorable, if the ***corresponding vertices*** in the PGKE coloring process are not the same color.

Proof: It is given that a planar graph G exists. Let this planar graph be connected. Use *Planar Graph K_4 Expansion* on G to obtain the graph G'. By the *Planar Graph K_4 Expansion Coloring Lemma* we know that G' is 4 – colorable.

Now that G' is properly 4 – colored, remove all the additional edges that were added in the *Planar Graph K_4 Expansion* process. This will provide the original graph G and since removing edges will not change the coloring of a graph, G is properly 4 – colored.

If G is not a connected graph, then execute the above steps on each connected component and the same result will hold. ■

Results

Unfortunately, the results of this paper were not what was expected upon initiating this research. The goal was to provide a written proof for the famous 4 – Color Theorem. What resulted was a weaker version of the 4 – Color Theorem.

One good that that came from this writing is that it is set up nicely to actually prove the 4 – Color Theorem, pending a smaller proof of corollary 1.1 without the exclusions that are made in the statement of the corollary. If one can show that dependent sets of vertices must always be in two K_4 that intersect at the three remaining vertices, then the paper is easily adjustable to prove the entire 4 – Color Theorem.

Case 2 in the **Planar Graph K_4 Expansion Coloring Lemma** is easily proven if one can prove the above fact. Everything else follows from there. The "punch line" result does not even have to change. While that seems like a daunting task, it seems very likely that the result is true. Now, it only requires a creative method of proving it.

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