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Transfinite Ordinal Arithmetic

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TRANSFINITE ORDINAL ARITHMETIC

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Abstract: Following the literature from the origin of Set Theory in the late 19th century to more current times, an arithmetic of finite and transfinite ordinal numbers is outlined. The concept of a set is outlined and directed to the understanding that an ordinal, a special kind of number, is a particular kind of well-ordered set. From this, the idea of counting ordinals is introduced. With the fundamental notion of counting addressed: then addition, multiplication, and exponentiation are defined and developed by established fundamentals of Set Theory. Many known theorems are based upon this foundation. Ultimately, as part of the conclusion, a table of many simplified results of ordinal arithmetic with these three operations are presented.

Table of Contents

List of Figures

> *To find the principles of mathematical being as a whole, we must ascend to those all-pervading principles that generate everything from themselves: namely the Limit* [Infinitesimal Monad] *and the Unlimited* [Infinite Dyad]. *For these, the two highest principles after the indescribable and utterly incomprehensible causation of the One, give rise to everything else, including mathematical beings.*
>
> Proclus[1]

The study and contemplation of infinity dates back to antiquity. We can identify forms of infinity from at least as early as the Pythagoreans and the Platonists. Among the earliest of these ideas was to associate the infinite with the unbounded Dyad, in opposition to the bounded limit or Monad. Embedded in the very definition of a set are these very notions of unity and the division of duality, in the unified set and its distinct elements, respectively. Geometrically these were attributed to the line and point. Theologically the Finite Monad was attributed to God, as the limit and source of all things, and to the Infinite was often associated the Devil, through the associated division and strife of Duality.[2] We could muse about the history of controversy and argumentation surrounding the infinite being perhaps more than mere coincidence.

We further see the concept of the infinite popping up throughout history from the paradoxes of Zeno to the infinitesimal Calculus of Leibniz and Newton. We find a continuity of development of our relationship with the infinite, which pervades Western intellectual culture. However, as long as our dance with the infinite has gone on, it has ever been met with a critical eye. This is perhaps the echoes of the Pythagorean legend of the traitor who discovered and/or revealed the concept of the incommensurables which poked holes in the perfection of Pythagorean thought. The same sort of minds that couldn't recognize the

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[1] Proclus. *A Commentary on the First Book of Euclid's Elements*. p 4. My addition in brackets.
[2] Iamblichus. *The Theology of Arithmetic*.

genius of Leibniz's infinitesimals as a basis for the Calculus, to replace it with cumbersome limit symbolism, only for the idea of infinitesimals to find new footing in the 20th century through the work of Abraham Robinson. [3] It seems that finitists are only willing to accept that which they can actually see, but as Giordano Bruno would argue a few centuries before Cantor:

> *It's not with our senses that we may see the infinite; the senses cannot reach the conclusion we seek, because the infinite is not an object for the senses.* [4]

In other words, the infinite is accessible only to the mind.

It is only in last century or so that the infinite really started to have rigorous mathematical foundation that is widely accepted – and that is the fruit of Georg Cantor's brainchild. While Georg Cantor's theory of Transfinite Numbers certainly didn't appear out of a vacuum, we largely owe our modern acceptance of the infinite as a proper, formal mathematical object, or objects, to him and his Set Theory. In 1874, Cantor published his first article on the subject called "On a Property of the Collection of All Real Algebraic Numbers." By 1895, some 20 years later at the age of 50, he had a well-developed concept of the Transfinite Numbers as we see in his *Contributions to the Founding of the Theory of Transfinite Numbers*.[5]

While Cantor's symbolism would in some cases seem somewhat foreign to a student of modern Set Theory, it is his conceptual foundation and approach to dealing with the infinite that has carried on into today. Set Theory has found its way into seemingly every

[3] Robinson, Abraham. *Non-standard Analysis*.
[4] Bruno, Giordano. *On the Infinite, the Universe, & the Worlds*. "First Dialogue." p 36.
[5] Cantor, Georg. *Contributions to the Founding of the Theory of Transfinite Numbers*.

area of mathematics and seems to unify most, if not all, of modern mathematics by the use of common symbolism this area of study.

Set Theory and Sets

While Cantor is credited as the father of Set Theory, the notion of sets certainly predates Cantor. We see sophisticated writings on the subject from the likes of Bernard Balzano, whose work had significant influence upon the work of Cantor. Consistent with Cantor, Balzano defined a set as "an aggregate of well-defined objects, or a whole composed of well-defined members."[6] In the strict sense of the word, a set can be a collection of anything real, imagined, or both. So we could have a set of pieces of candy in a bag. We could also have the set of the fantasy creatures of a phoenix, a dragon, and a unicorn. We could also have the set containing a pink elephant, the number two, and the planet Jupiter. However, in Set Theory, we generally are observing specific kinds of sets of mathematical objects.

Ordinals and Cardinals Defined

The two primary types of number discussed in Set Theory are called ordinals and cardinals. While cardinal numbers are not the focus of the present paper, it is still of value to observe its character in contrast with ordinal numbers, to get a clearer sense of the nature of the latter. Cardinal numbers are also much more quickly understood. A cardinal number, in its simplest sense, is just how many of something there is. So if I have the set of five athletes running a race, the cardinality of that set of people is 5. If we are discussing the fifth person to complete the race, this corresponds to an ordinal of 5, and it implies that four came

[6] Balzano, Bernard. *Paradoxes of the Infinite*. p 76.

before it. So both of these relate to the number 5. This becomes less immediately intuitive from our general sense of number when we jump into looking at transfinite numbers. Ordinality has more to do with how a set is organized, or ordered.

We build up our concept of the pure sets based first upon the *Axiom of the Empty Set*, which states simply that there exists an empty set, symbolically:

$$\exists A[\forall x, x \notin A].$$

In other words, there exists some set A such that for any and every object x, x is not in A. We denote the empty set as empty set brackets { } or as $\emptyset$.

To construct the remainder of the ordinals, we will require another one of the Axioms of ZFC, namely the *Axiom of Infinity*. This axiom states the existence of a set that contains the empty set as one of its elements, and for every element of the set (the empty set being the first of such elements) there exists another element of the set that itself is a set containing that element. Symbolically this is represented as

$$\exists A\left(\emptyset \in A \wedge \forall x\left(x \in A \rightarrow \exists y\big(y \in A \wedge \forall z(z \in y \leftrightarrow z = x)\big)\right)\right).$$

So given the set contains the empty set $\emptyset$, it also contains the set containing the empty set in the form $\{\emptyset\}$. This newly discovered element is then further subject to the same clause, so our set A also contains $\big\{\{\emptyset\}\big\}$ as a member. This goes on ad infinitum, and looks something like

$$A = \left\{\emptyset, \{\emptyset\}, \big\{\{\emptyset\}\big\}, \Big\{\big\{\{\emptyset\}\big\}\Big\}, \bigg\{\Big\{\big\{\{\emptyset\}\big\}\Big\}\bigg\}, \Bigg\{\bigg\{\Big\{\big\{\{\emptyset\}\big\}\Big\}\bigg\}\Bigg\}, \left\{\Bigg\{\bigg\{\Big\{\big\{\{\emptyset\}\big\}\Big\}\bigg\}\Bigg\}\right\}, \ldots\right\}.$$

The set of ordinals can then be constructed from these two axioms paired with the concept of the subset. The ordinals are essentially specific combinations of elements of the set guaranteed to exist by the above *Axiom of Infinity*. Per the definition of an ordinal by

John von Neumann in 1923: "Every ordinal is the set of all ordinals that precede it."[7] So the first ordinal is the empty set $\emptyset$. The second is the set containing the previous ordinal, so it looks like $\{\emptyset\}$. The third is the set containing all the previous ordinals, so $\{\emptyset, \{\emptyset\}\}$. The fourth gets a little messier and is a set containing all the previous ordinals like $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. This is clearly not a convenient form of notation. Next will be outlined a more practical form of symbolism, followed by a formal definition of an ordinal number.

So instead of the cumbersome sets within sets within sets within sets, etc., we will give each ordinal a name, or label. This custom of giving each ordinal a label from the non-negative integers is generally credited to von Neumann. In his work, he uses the equals sign, and this custom has continued in literature for the last century, including more recent publications such as Keith Devlin's *The Joy of Sets*.[8] Some modern mathematicians have adopted symbolism other than the simple equals sign for sake of clarity. Here will use the delta-equal-to symbolism ($\triangleq$), used by Takeuti in *Introduction to Axiomatic Set Theory*, to set up our definitions, and thereafter will assume the definitions hold.[9] So the empty set is defined to be represented by the symbol 0, the set containing the empty set will be 1, etc.

$$0 \triangleq \emptyset$$

$$1 \triangleq \{\emptyset\} = \{0\}$$

$$2 \triangleq \{\emptyset, \{\emptyset\}\} = \{0,1\}$$

$$3 \triangleq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0,1,2\}$$

$$4 \triangleq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} = \{0,1,2,3\}$$

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[7] von Neumann, John. "On the Introduction of Transfinite Numbers." p 347.
[8] Devlin, Keith. *The Joy of Sets*.
[9] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory*. p 41.

$$5 \triangleq \{0,1,2,3,4\}$$

$$6 \triangleq \{0,1,2,3,4,5\}$$

Cantor defined ordinals as follows.

> *Every simply ordered aggregate M has a definite ordered type $\overline{M}$; this type is the general concept which results from M if we abstract from the nature of its elements while retaining their order of precedence, so that out of them proceed units which stand in a definite relation of precedence to one another.* [10]

More formally, an ordinal is a well-ordered set such that every element less than the ordinal is an element of that set. Symbolically an ordinal α is defined to be

$$\alpha \triangleq \{x \in X | x < \alpha\},$$

where X is the set of all ordinals.[11] For any two ordinals α and β, if $\alpha < \beta$ then $\alpha \in \beta$, and if $\alpha \in \beta$ then $\alpha < \beta$. The less than relation and the member relation are interchangeable. In this paper, the less than relationship will be used predominantly.

A successor ordinal is the ordinal that follows any given ordinal. The successor of an ordinal α is often denoted α^+, but we will use the symbolism more convenient for arithmetic of the form $\alpha^+ = \alpha + 1$. The successor of the ordinal α is defined symbolically as

$$\alpha + 1 \triangleq \alpha \cup \{\alpha\}.$$

For example, we would intuitively think that the successor of 5 is 6, per our sense of integers. We will see this is also the case for our ordinals. Using the above notation, along with the knowledge that the ordinal $5 = \{0,1,2,3,4\}$, we find that

$$5 + 1 = 5 \cup \{5\} = \{0,1,2,3,4\} \cup \{5\} = \{0,1,2,3,4,5\} = 6.$$

[10] Cantor, Georg. *Contributions to the Founding of the Theory of Transfinite Numbers*. pp 151-152.
[11] Devlin, Keith. *The Joy of Sets*. p 66.

A transfinite number is one that is "beyond infinite." The first transfinite ordinal is the set of all finite ordinals as well as their supremum, symbolically $\omega \triangleq \{0,1,2,\dots\}$. This first transfinite ordinal also has a successor $\omega+1$, the second transfinite ordinal. By the above definition of a successor ordinal:

$$\omega + 1 = \omega \cup \{\omega\} = \{0,1,2,\dots,\omega\}.$$

Note that in this paper, for the sake of simplicity, when we add a successor ordinal $\alpha+1$ like $\beta+(\alpha+1)$ that we will generally refer to this as $\beta+\alpha+1$. To denote the successor of a sum, we will always use parenthesis to denote that we are doing so. Thus the successor of the ordinal $\alpha+\beta$ will be denoted as $(\alpha+\beta)+1$.

Every ordinal has a successor ordinal. Not all ordinals have an immediate predecessor. With the exception of 0, any ordinal that doesn't have an immediate predecessor is called a limit ordinal. If an ordinal α is a limit ordinal, we denote this as $\alpha \in K_{II}$, where K_{II} is the class of all limit ordinals. Symbolically we can say

$$\alpha \neq 0 \wedge \nexists\beta[\beta+1=\alpha], \forall\alpha \in K_{II}.$$

If an ordinal β is not a limit ordinal, therefore it (with the exception of 0, which is also a nonlimit ordinal) has an immediate predecessor, then we denote this as $\beta \in K_I$. Symbolically we can say

$$\alpha = 0 \vee \exists\beta[\beta+1=\alpha], \forall\alpha \in K_I.$$

All ordinals fall into one of these two classes. Note that K_I and K_{II} are not ordinals. Therefore, it is not appropriate to say something like $\alpha < K_{II}$ in place of $\alpha \in K_{II}$.

The first limit ordinal ω is also our first transfinite ordinal. The ordinal ω is the supremum of all finite ordinals, and has as its member all finite ordinals, symbolically

$$\omega \triangleq \{0,1,2,\dots\}.$$

Notice this set has no last element, and thus ω has no immediate predecessor, which is the definition of a limit ordinal. There is no ordinal α such that $\alpha + 1 = \omega$. If there were, we would have a contradiction. As every element of ω is finite, if there were some element α that was an immediate predecessor of ω, α being a member of ω would imply that α is finite. This would then imply that ω, being the successor of a finite ordinal, would therefore be finite. There we have the desired contradiction.

One of the more interesting results of Set Theory is that it shows the existence of different kinds of infinity. We see this happen both in the case of the present subject of ordinals as well as with the further development of cardinals. While we must give credit to Cantor for these conclusions and their proof, the idea itself was not an original one, as we see from the writing of Bernard Balzano who died three years after the birth of Cantor.

> *Even in the examples of the infinite so far considered, it could not escape our notice that not all infinite sets can be deemed equal with respect to the multiplicity of the members. On the contrary, many of them are greater (or smaller) than some other in the sense that the one includes the other as part of itself (or stands to the other in the relation of part to the whole). Many consider this as yet another paradox, and indeed, in the eyes of all who define the infinite as that which is incapable of increase, the idea of one infinite being greater than another must seem not merely paradoxical, but even downright contradictory.* [12]

Transfinite Induction and Recursive Definitions

As transfinite ordinals often behave differently than finite ordinals, we have to use methods of proof appropriate to transfinite ordinals to properly deal with them. A common method, which is used heavily in this paper, is called Transfinite Induction. The structure, outlined by Suppes,[13] is done in three parts as follows. As it turns out, most often we are doing induction on γ, so the following outline is in terms of induction on γ for ease in

[12] Balzano, Bernard. *Paradoxes of the Infinite*. p 95.
[13] Suppes, Patrick. *Axiomatic Set Theory*. p 197.

translation. In the case of each proof using this schema, each portion will be appropriately labeled at **Part 1**, **Part 2**, and **Part 3**.

Part 1 is to demonstrate that the hypothesis works for the base case. Generally this is done with $\gamma = 0$. As we will see, this is not always the case, particularly when 0 isn't an option due to the way the ordinal in question is defined.

Part 2 is akin to weak induction. We assume the hypothesis holds for γ, and show it holds for the case of the successor ordinal $\gamma + 1$.

Part 3 is akin to strong induction. We assume that γ is a limit ordinal, symbolically $\gamma \in K_{II}$. We further assume the hypothesis holds for all elements of γ. With this in mind, we then demonstrate that the hypothesis holds for γ.

As all ordinals fall into the category of either a limit ordinal or not a limit ordinal, if the hypothesis holds for all three parts, then the hypothesis holds for all ordinals. Generally Set Theory theorems are only presented that hold for all ordinals, from both classes of K_I and K_{II}, so that the theorem always work no matter the ordinals being used.

Similar in form to Transfinite Induction, we will use Transfinite Recursion to lay out some definitions as fundamental standards of behavior for our arithmetical operations. The three parts of these recursive definitions are essentially the same as those of the induction process. The difference is that instead of proving, we take the statements as fundamental truths, without proof.

1. Ordinal Addition

Now that we have a sense of what ordinals are and how to count them, we will introduce the first arithmetical operation – addition. First will be shown the simple, intuitive conceptions of addition, and then a very formal definition. Finally, for practical purposes, addition will be codified into a recursive definition.

Let us start with a general sense of what it means to add two ordinals. If we add two ordinals like $\alpha + \beta$, this means that we count α number of times, and then we count β more times afterward. The order in which these are added makes a difference, particularly when working with transfinite ordinals. Assuming α and β are finite ordinals, the sum can be represented in the following visual, intuitive way (note: the ellipses the following diagram imply an ambiguous number of terms, and does not imply counting without end as it generally does in Set Theory). If α and β are finite ordinals then $\alpha = \{0,1,2, \dots, \alpha - 1\}$ and $\beta = \{0,1,2, \dots, \beta - 1\}$, where $\alpha - 1$ is the immediate predecessor of α, and $\beta - 1$ is the immediate predecessor of β. Figure 1 illustrates this rather intuitively.

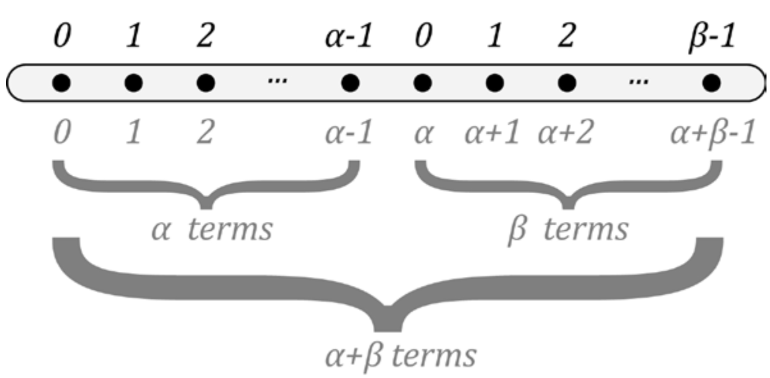

Figure 1. Visualization of $\alpha + \beta$

Formal Definition of Addition Using the Cartesian Product

John von Neumann, while not using that specific phrase, introduced the concept of *order type* in 1923. Given any well-ordered set A, we define a mapping $f(x)$ of each element of that set to an ordinal. In this way then, the set A itself inherits the quality of being an ordinal. If a_0 is the first element of A, then we map that to the first ordinal 0. The second element a_1 is mapped to the second ordinal 1. For a concrete example, let us suppose then that $A = \{a_0, a_1, a_2, a_2\}$ and is well-ordered by $a_0 < a_1 < a_2 < a_3$, then

$f(a_0) \triangleq 0$,

$f(a_1) \triangleq 1$,

$f(a_2) \triangleq 2$, and

$f(a_3) \triangleq 3$.

Therefore $A \triangleq \{0,1,2,3\} = 4$, and A is said to have *order type* of 4.[14]

The formal definition of ordinal addition is presented as a Cartesian product, well-ordered in a specific way to be defined. We define the sum of two ordinals α and β to be

$$\alpha + \beta \triangleq \overline{\alpha \times \{0\} \cup \beta \times \{1\}}.$$

This overbar means that the sum of α and β is the *order type* (given the well-ordering defined below) of the union set of the cross products of $\alpha \times \{0\}$ and $\beta \times \{1\}$. The values 0 and 1 in the given definition above need only be such that $0 < 1$, and not all texts use the same values. It would be just as good to say that

$$\alpha + \beta \triangleq \overline{\alpha \times \{m\} \cup \beta \times \{n\}}, \qquad \textit{such that } m < n.$$

[14] von Neumann, John. "On the Introduction of Transfinite Numbers." p 348.

However, for practical purposes of adding two values, we will just stick with the 0 and 1. This has to be ordered in a very specific way to get consistent results, so we say that for two ordered pairs (u, x) and (v, y):

$$(u, x) < (v, y) \ \ if \ \ x < y$$

and

$$(u, x) < (v, y) \ \ if \ x = y \ \ and \ u < v.$$

So for the ordered pairs (3,5) and (1,7), we see that $(3,5) < (1,7)$ as $5 < 7$. For the ordered pairs (6,2) and (9,2), we see that $(6,2) < (9,2)$ as $6 < 9$ and the second terms are the same.

Let us also consider a set where there are missing elements. For example, we define the ordinal $6 = \{0,1,2,3,4,5\}$. In order for this set to actually be equal to 6, it must contain all ordinals less than 6. Clearly this is not the case with the set $\{1,2,3,4,5\}$. However, it is still well-ordered such that each element that is less than another element is still a member of that element. So we must further define order type. The *Order Type* of a well-ordered set A, denoted $\bar{A}$, is defined to be the smallest ordinal isomorphic to A. Any two well-ordered sets with the same order type represent the same ordinal. So in the case of the set $\{1,2,3,4,5\}$, it is isomorphic to $\{0,1,2,3,4\} = 5$. Therefore the set of ordinals $\{0,1,2,3,4\}$ is said to have an order type of 5.

As an example, the sum of the two values 2 and 3 is

$$2 + 3 = \overline{2 \times \{0\} \cup 3 \times \{1\}}$$

Recalling that the ordinal $2 = \{0,1\}$ and the ordinal $3 = \{0,1,2\}$, we have

$$= \overline{\{0,1\} \times \{0\} \cup \{0,1,2\} \times \{1\}}$$

$$= \overline{\{(0,0), (1,0), (0,1), (1,1), (2,1)\}}.$$

This set is isomorphic to $\{0,1,2,3,4\} = 5$, and therefore $2 + 3 = 5$.

So this is simple enough, and this matches with our intuition about adding whole numbers. In the case of adding finite ordinals, the particular ordering defined above doesn't really affect the resulting sum. For finite ordinals, we could order the union set in any way we like and still have the same result. The particular ordering defined above really shows its worth when we starting adding transfinite ordinals. It will now be shown that $\omega + 2 \neq 2 + \omega$, and thus motivate why the order in which we add ordinals has to be clearly defined.

Let us start by looking at the sum of 2 and ω:

$$2 + \omega = \overline{2 \times \{0\} \cup \omega \times \{1\}}.$$

Recalling that $2 = \{0,1\}$ and $\omega = \{0,1,2,\dots\}$, we then have the statement

$$= \overline{\{0,1\} \times \{0\} \cup \{0,1,2,\dots\} \times \{1\}}.$$

Then by taking the Cartesian product we have

$$= \overline{\{(0,0),(1,0),(0,1),(1,1),(2,1),\dots\}}.$$

This set is isomorphic to $\{0,1,2,\dots\} = \omega$, and thus

$$2 + \omega = \omega.$$

Let's continue by looking at the sum of ω and 2:

$$\omega + 2 = \overline{\omega \times \{0\} \cup 2 \times \{1\}}$$

$$= \overline{\{0,1,2,\dots\} \times \{0\} \cup \{0,1\} \times \{1\}}.$$

Then by taking the Cartesian product we have

$$= \overline{\{(0,0),(1,0),(2,0),\dots,(0,1),(1,1)\}}.$$

The striking difference at this point, as compared with the previous example, is where the ellipses fall. The ellipses, in the context of set theory anyway, just imply counting for forever. In the previous example, they occurred at the tail end of the set notation. Here, however, they fall somewhere before the end. So this set would be isomorphic to $\{0,1,2,\ldots,\omega,\omega+1\}$, which is greater than ω. Thus $2+\omega \neq \omega+2$.

So we see here there are values beyond our first conception of infinity. In some respect, it seems like we haven't really captured the infinite. As soon as we apply a metric to it (in this case ω), the nature of the infinite immediately defies this metric and show us there is now something more, something greater: $\omega+1$. It is unlikely we will ever find the largest infinity, as the infinite is by definition beyond measure and unbounded. As Plutarch wrote:

I am all that has been, and is, and shall be, no one has yet raised my veil.[15]

It is helpful for the development of an intuition about the behavior of counting and addition to have a visual representation. The *supertask* is an excellent sort of representation for a visual representation of infinity. It is based upon the philosophy of Zeno of Elea, famously known for his paradoxes about the infinite. James F. Thomson, writes of the *supertask* in a more modern symbolic-logic version of Zeno's Dichotomy Paradox (see Figure 2):

> *To complete any journey you must complete an infinite number of journeys. For to arrive from A to B you must first go from A to A', the mid-point of A of B, and thence to A'', the mid-point of A' and B, and so on. But it is logically absurd that someone should have completed all of an infinite number of journeys, just as it is logically absurd that someone should have completed all of an infinite number of tasks. Therefore, it is absurd that anyone has ever completed any journey.*[16]

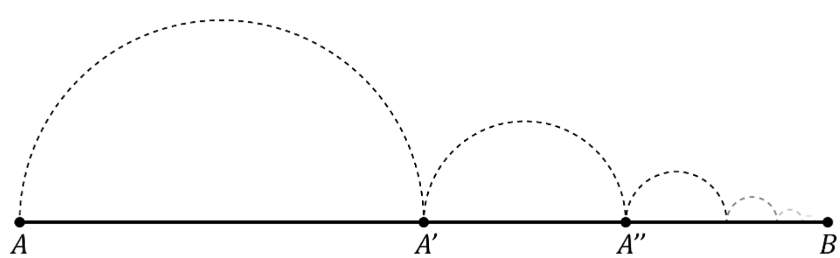

Figure 2 - Visualization of Zeno's Dichotomy Paradox

[15] Plutarch. *Isis and Osiris*.
[16] Benacerraf, Paul. *Tasks, Super-Tasks, and the Modern Eleatics*. p 766.

Thomson uses the notion of the supertask in an argument about a lamp (*which is strikingly similar to the dilemma of final term in an infinite alternating product, discussed in the afterthoughts of Theorem 3.5*):

> *There are certain reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and your press the button, the lamp goes off. So if the lamp was originally off and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half minute, and so on… After I have completed the whole infinite sequence of jabs, i.e. at the end of the two minutes, is the lamp on or off?...It cannot be on, because I did not ever turn it on without at least turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.*[17]

Thomson used this model to show the absurdity of physical objects engaged in an infinite number of tasks (which cooperates with Bruno's assertion that the infinite is not for the senses). However, the mathematical philosopher Paul Benacerraf adopted the model as logically possible (to the realm of the intelligible, the Platonic world opposite that of the sensible) and applied to counting to infinity or "what happens at the ωth moment".[18]

Imagine an infinite number of fence posts, or vertical lines, that are all equidistant. Obviously in this sense, it is not directly observable. Put the fence posts into a geometric proportion converging to zero, and the infinite is thus represented. Figure 3 is such a visual representation of counting to infinity, a representation of the ordinal ω.

[17] Benacerraf, Paul. *Tasks, Super-Tasks, and the Modern Eleatics*. pp 767-768.
[18] Ibid. p 777.

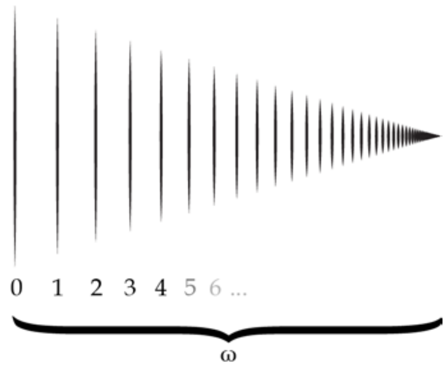

Figure 3. Visualization of ω

Recall that $\omega \triangleq \{0,1,2,\ldots\}$ and that ω is not a part of this set. For that matter, if it isn't already clear, no ordinal is a member of itself. So were we to count all the way to infinity, the next ordinal would be omega. The value following ω is $\omega + 1$. Recall from the definition of a successor ordinal, and by the fact that $\omega = \{0,1,2,\ldots\}$, that

$$\omega + 1 = \omega \cup \{\omega\} = \{0,1,2,\ldots,\omega\}.$$

Figure 4 is a representation of these next two ordinals, in the mix of the ordinal $\omega + \omega$, which we will in the section on multiplication refer to as $\omega 2$.

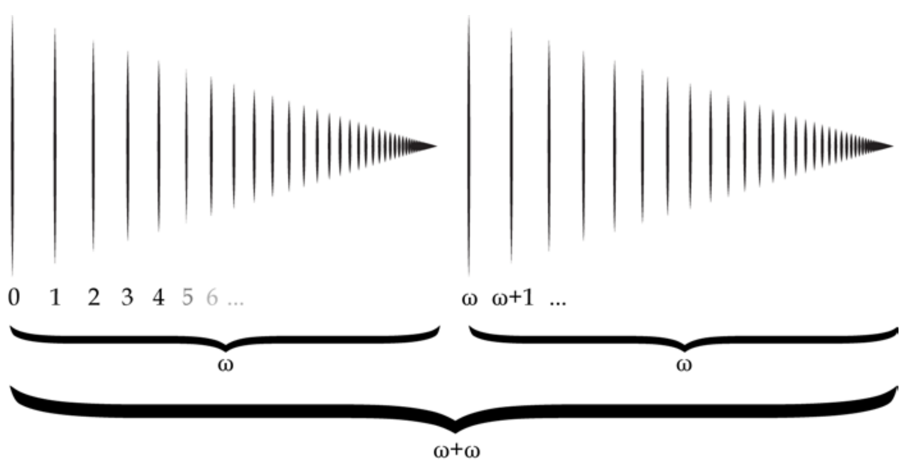

Figure 4. Visualization of $\omega + \omega$

Recursive Definition of Addition

As the above symbolism using the Cartesian product is cumbersome, it is more practical to have a simple definition of addition that will work for us in any situation. To make this work, we set up a recursive definition of addition, based upon the number which we are adding on the right hand side. The following is adapted from Patrick Suppes' *Axiomatic Set Theory*.[19]

Recursive Definition of Ordinal Addition

1. $\alpha + 0 \triangleq 0 + \alpha \triangleq \alpha$.

   Adding zero on the right (as well as on the left) serves as the additive identity.

2. $\alpha + \beta + 1 \triangleq (\alpha + \beta) + 1$.

   Adding the successor ordinal $\beta + 1$ to an ordinal α is the same as the successor of the sum of $\alpha + \beta$.

3. When β is a limit ordinal

$$\alpha + \beta \triangleq \bigcup_{\gamma<\beta} \alpha + \gamma.$$

   In other words, the supremum of all $\alpha + \gamma$, where $\gamma < \beta$, is $\alpha + \beta$. This will be used a lot in the third part of the transfinite induction process.

Results and Proofs

Now that we have a sense of what an ordinal is, and some fundamental notions of counting and adding on a small scale, we will explore some of the consequences of these premises. The proofs that follow, where cited, are adapted from a study of Patrick Suppe's

[19] Suppes, Patrick. *Axiomatic Set Theory*. p 205.

Axiomatic Set Theory and Gaisi Takeuti and Wilson Zaring's *Introduction to Axiomatic Set Theory*. We will begin with a proof of right monotonicity when adding ordinals.

<u>***Theorem* 1.0: $\beta < \gamma \rightarrow \alpha + \beta < \alpha + \gamma$**</u>

For any three ordinals α, β and γ: if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

<u>Proof</u>. [20,21] This will be proven by transfinite induction on γ.

Part 1. We will start with the base case where $\gamma = 0$. As 0 is the empty set and the smallest ordinal,

$\beta < 0$ is false, and the theorem holds vacuously.

Part 2. Assume the hypothesis is true, and then show that $\beta < \gamma + 1$ implies that $\alpha + \beta < \alpha + \gamma + 1$. Start with the assumption that $\beta < \gamma + 1$, and then we have the implication $\beta \leq \gamma$, as the stated assumption is true in either case of $\beta = \gamma$ or $\beta < \gamma$. If $\beta = \gamma$, then clearly $\beta < \beta + 1 = \gamma + 1$. Otherwise, β must be smaller than γ for this to be true. In other words, $\beta + \eta = \gamma$, for some ordinal $\eta > 0$. Then we have two cases: either $\beta < \gamma$ or $\beta = \gamma$.

In the case of $\beta < \gamma$, per our hypothesis

$$\alpha + \beta < \alpha + \gamma.$$

Because any ordinal is less than its successor:

$$\alpha + \gamma < (\alpha + \gamma) + 1.$$

By the recursive definition of Ordinal Addition $(\alpha + \gamma) + 1 = \alpha + \gamma + 1$, and so we have:

$$\alpha + \beta < \alpha + \gamma < \alpha + \gamma + 1$$

$$\rightarrow \alpha + \beta < \alpha + \gamma + 1.$$

[20] Suppes, Patrick. *Axiomatic Set Theory*. p 207.
[21] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory*. p 58.

In the second case, we have $\beta = \gamma$. Then $\beta < \gamma + 1$ and $\alpha + \beta = \alpha + \gamma$. Again, because any ordinal is less than its successor:

$$\alpha + \beta = \alpha + \gamma < \alpha + \gamma + 1.$$

Part 3. We will view the case in which γ is a limit ordinal, and assume the hypothesis holds for all elements of γ such that

$$(\forall\delta)[\beta < \delta < \gamma \rightarrow \alpha + \beta < \alpha + \delta].$$

With all this in mind, will show that the hypothesis holds for γ.

Per the defined relationship between β, δ, and γ, certainly $\alpha + \beta < \alpha + \delta$. Further, it is also certainly true that $\alpha + \delta$ is less than or equal to the supremum of all ordinals of the form $\alpha + \delta$, which is equal to $\alpha + \gamma$. Thus

$$\alpha + \beta < \alpha + \delta \leq \bigcup_{\delta<\gamma} \alpha + \delta = \alpha + \gamma.$$

■

<u>$\boldsymbol{Theorem\ 1.1}$: $\boldsymbol{\beta \in K_{II} \rightarrow (\alpha + \beta) \in K_{II}}$</u>

If β is a limit ordinal, then $\alpha + \beta$ is also a limit ordinal. In other words, if we add a limit ordinal on the right of any ordinal, then the result is always a limit ordinal.

<u>Proof</u>. [22,23] This will be proven by contradiction. We will suppose $\alpha + \beta$ is not a limit ordinal, and reach our desired contradiction. Because β is a limit ordinal and $\alpha + \beta \geq \beta$ for all ordinals α and β, we can conclude:

$$\alpha + \beta \neq 0$$

[22] Suppes, Patrick. *Axiomatic Set Theory*. p 209.
[23] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 60.

Now because $\alpha + \beta$ is not a limit ordinal, per our assumption, that means it has a predecessor γ such that:

$$\gamma + 1 = \alpha + \beta.$$

While we have assumed that $\alpha + \beta$ is not a limit ordinal, we still hold β to be a limit ordinal, and so by the definition of adding a limit ordinal:

$$\alpha + \beta = \bigcup_{\delta<\beta} (\alpha + \delta).$$

Now because $\gamma < \gamma + 1$ and $\gamma + 1 = \alpha + \beta$, we can say that $\gamma < \alpha + \beta$, and thus:

$$\gamma < \bigcup_{\delta<\beta} (\alpha + \delta).$$

If $\alpha + \beta$ is not a limit ordinal, then there is some $\delta_1 < \beta$, such that $\gamma < \alpha + \delta_1$. Thus

$$\alpha + \beta = \gamma + 1 < (\alpha + \delta_1) + 1 = \alpha + \delta_1 + 1,$$

which yields

$$\alpha + \beta < \alpha + \delta_1 + 1.$$

Since $\delta < \beta$ and β is a limit ordinal, then $(\delta + 1) < \beta$. So then

$$\gamma + 1 < \bigcup_{\delta<\beta} (\alpha + \delta) = \gamma + 1.$$

So $(\gamma + 1) < (\gamma + 1)$, which is false, as no ordinal is less than itself. Therefore we conclude that

$\alpha + \beta \in K_{II}$.

■

Theorem 1.2: $(\boldsymbol{\alpha}+\boldsymbol{\beta})+\boldsymbol{\gamma}=\boldsymbol{\alpha}+(\boldsymbol{\beta}+\boldsymbol{\gamma})$

This theorem states that ordinal addition is associative, and holds true, no matter the ordinals we are adding; all three ordinals in this case can be either limit ordinals or not limit ordinals.

Proof.[24,25] This will be proven by induction on γ.

Part 1. Start with the base case, where $\gamma = 0$. It just needs to be shown that the hypothesis holds in this case, and so we freely substitute γ for 0, and vice versa, as such:

$$(\alpha+\beta)+\gamma=(\alpha+\beta)+0=\alpha+\beta=\alpha+(\beta+0)=\alpha+(\beta+\gamma).$$

Part 2. We will then assume our hypothesis and show it holds for the successor, such that $(\alpha+\beta)+\gamma+1=\alpha+(\beta+\gamma+1)$. By the recursive definition of addition of ordinals:

$$(\alpha+\beta)+\gamma+1=\big((\alpha+\beta)+\gamma\big)+1.$$

By our assumed hypothesis we can further state:

$$\big((\alpha+\beta)+\gamma\big)+1=\big(\alpha+(\beta+\gamma)\big)+1.$$

Again, but the recursive definition of addition, we can take two more steps and find our desired result:

$$\big(\alpha+(\beta+\gamma)\big)+1=\alpha+(\beta+\gamma)+1=\alpha+(\beta+\gamma+1).$$

Thus $(\alpha+\beta)+\gamma+1=\alpha+(\beta+\gamma+1)$, as desired.

Part 3. We will deal with the case in which γ is a limit ordinal, symbolically as $\gamma \in K_{II}$. We will also assume that the hypothesis holds for all the elements of γ, so that

$$(\alpha+\beta)+\delta=\alpha+(\beta+\delta), \forall\delta<\gamma,$$

and then we will show it holds for γ.

[24] Suppes, Patrick. *Axiomatic Set Theory*. p 211.
[25] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 61.

By definition of adding a limit ordinal on the right:

$$(\alpha+\beta)+\gamma=\bigcup_{\delta<\gamma}(\alpha+\beta)+\delta.$$

By our hypothesis:

$$\bigcup_{\delta<\gamma}(\alpha+\beta)+\delta=\bigcup_{\delta<\gamma}\alpha+(\beta+\delta).$$

By Theorem 1.1 we know that if γ is a limit ordinal, then $\beta+\gamma$ is also a limit ordinal, and so

$$\alpha+(\beta+\gamma)=\bigcup_{\eta<\beta+\gamma}\alpha+\eta.$$

It is thus given that $\eta<\beta+\gamma$, where η represents all values less than $\beta+\gamma$. Since $\delta<\gamma$, and δ represents all values less than γ, we can also conclude that given any η, there exists some δ such that $\eta=\beta+\delta$. Thus

$$\alpha+(\beta+\delta)\leq\alpha+\eta.$$

Alternatively, if we begin solely with the defined value that $\eta<\beta+\gamma$, then either $\eta<\beta$ or there must exist some δ such that $\eta=\beta+\delta$. First suppose that $\eta<\beta$. Then certainly $\alpha+\eta<\alpha+\beta$ per Theorem 1.0. Given we can expand our conditions to include equality, and that adding nothing to β is still β:

$$\alpha+\eta\leq\alpha+(\beta+0).$$

Because γ is a limit ordinal, we know that $\gamma\geq\omega>0$, and so $a+(\beta+0)<\alpha+(\beta+\gamma)$ and

$$\alpha+\eta\leq\alpha+(\beta+\gamma).$$

Otherwise $\eta=\beta+\delta$, which implies that $\alpha+\eta=\alpha+(\beta+\delta)$, and so

$$\alpha+\eta\leq\alpha+(\beta+\delta).$$

So we have thus shown that $\alpha+(\beta+\delta)\leq\alpha+\eta\leq\alpha+(\beta+\delta)$ and thus

$$\alpha+\eta=\alpha+(\beta+\delta).$$

As $\alpha + \beta = \alpha + \gamma$ if an only if $\beta = \gamma$ we can state therefore that $\alpha + \eta = \alpha + (\beta + \delta)$ implies

$$\eta = \beta + \delta.$$

Therefore we can conclude

$$\bigcup_{\delta<\gamma} \alpha + (\beta + \delta) = \bigcup_{\eta<\beta+\gamma} \alpha + \eta$$

and

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

■

What is often given but a passing glance in texts on set theory is that ordinal addition is not commutative. It seems uncommon for a textbook on the subject to address things in depth that are not true. What I will show, though it is certainly not true in all cases, that it does hold in some cases. It is easy to demonstrate that ordinal addition is not always commutative by a counterexample. For example:

$$\omega + 2 > \omega = 2 + \omega.$$

However, there is at least one circumstance where commutativity would hold. Commutativity holds when we are considering adding strictly finite ordinals. We are going to prove that addition of finite ordinals is commutative for all finite ordinals, but we will start with a base case and first prove that $1 + n = n + 1$, where n is a finite ordinal.

Lemma 1.3: $\mathbf{1 + n = n + 1}$, $\forall \boldsymbol{n < \omega}$

Proof. This will be proven by induction on n. The case where $n = 0$ is trivial, so we will start with the base case of $n = 1$:

$$1 + n = 1 + 1 = n + 1.$$

For the inductive step, we will assume the hypothesis that $1 + n = n + 1$ and then show this leads to $(n + 1) + 1 = 1 + (n + 1)$. First, because ordinal addition is associative, we have

$$1 + (n + 1) = (1 + n) + 1.$$

By our hypothesis we then have

$$(1 + n) + 1 = (n + 1) + 1,$$

which was what we wanted to show.

■

<u>Theorem 1.4: Addition of Finite Ordinals is Commutative</u>

$$m + n = n + m, \qquad \forall m, n < \omega$$

As it is customary, m and n are used to express finite ordinals.

<u>Proof</u>. This will be proven by induction on n. The base case where $n = 1$ was just proven above in Lemma 1.3.

For the inductive portion, we will assume the hypothesis and show that this leads us to the conclusion that $(m + 1) + n = n + (m + 1)$. Let us begin by using the known conclusion that ordinal addition is associative and we see that

$$(m + 1) + n = m + (1 + n).$$

For any finite ordinal n, it is clear from Lemma 1.3 that $1 + n = n + 1$ and so we have

$$m + (n + 1) = (m + n) + 1.$$

Then by our hypothesis we have

$$(n + m) + 1 = n + (m + 1),$$

which is what we wanted to show.

■

Commutativity of addition doesn't always work for transfinite ordinals. As such, it isn't generally presented as a theorem. However, based on the Theorem 1.4, we can derive further situations where commutativity holds.

<u>Corollary 1.5: Addition of Finite Multiples of an Ordinal is Commutative</u>

$$\alpha m + \alpha n = \alpha n + \alpha m, \forall m, n < \omega$$

Meaning if we are adding multiples of any ordinal α, then addition is commutative.

<u>Proof</u>. By the distributive property (proven in Theorem 2.2 in the section on multiplication, and the proof of that theorem doesn't rely on this theorem) we have the following:

$$am + an = a(m + n).$$

By Theorem 1.4 we know that $n + m = m + n$ and thus

$$a(m + n) = \alpha(n + m).$$

And then by distribution:

$$\alpha(n + m) = \alpha n + \alpha m.$$

■

Lastly, we will discuss a situation that are important to understand for simplifying some statements. Our primary conclusion will be that if we add any transfinite ordinal α to any finite ordinal m then

$$m + \alpha = \alpha.$$

Let us begin by recalling an earlier conclusion that $2 + \omega = \omega$. Remember that this sum looks like

$$\overline{\{(0,0), (1,0), (0,1), (1,1), (2,1), \dots\}}$$

which has an order type of ω. This happens no matter the finite value we start with.

Theorem 1.6: $\boldsymbol{m + \omega = \omega, \ \forall m < \omega}$

Proof. This will be proven by (not transfinite) induction on m.

We will start with the base case. The case where $m = 0 \rightarrow 0 + \omega = \omega$ is trivial and is already covered by the recursive definition of addition. The case of $m = 1$ is more important for our immediate purposes. To be clear, by the use of the cross product for finding a sum

$$1 + \omega = \overline{1 \times \{0\} \cup \omega \times \{1\}} = \overline{\{(0,0), (0,1), (1,1), (2,1), \dots\}} = \omega.$$

So the base case holds and $1 + \omega = \omega$.

For the induction, we will then assume that $m + \omega = \omega$ and show it leads to the conclusion that $(m + 1) + \omega = \omega$. By Theorem 1.2 ordinal addition is associative and so

$$(m + 1) + \omega = m + (1 + \omega).$$

We just showed that $1 + \omega = \omega$, and so

$$m + (1 + \omega) = m + \omega.$$

Finally, by our hypothesis

$$m + \omega = \omega.$$

Thus have our desired result:

$$(m + 1) + \omega = m + \omega = \omega.$$

■

A further important result follows from Theorem 1.6. Essentially if we add any transfinite ordinal to a finite ordinal, the resulting sum is the transfinite ordinal that we added.

Corollary 1.7: $\boldsymbol{m+\alpha=\alpha,\ \forall m<\omega,\ \alpha\geq\omega}$

Proof. Any ordinal α greater than or equal to ω can be written as $\alpha = \omega + \beta$, where β can be any ordinal. Thus

$$m + \alpha = m + (\omega + \beta).$$

Because ordinal addition is associative we know that

$$m + (\omega + \beta) = (m + \omega) + \beta.$$

From Theorem 1.6, we know that $m + \omega = \omega$ and thus

$$(m + \omega) + \beta = \omega + \beta = \alpha,$$

which was what we wanted.

■

2. Ordinal Multiplication

Our understanding of multiplication of ordinals is very similar to our elementary notion of multiplication of whole numbers, in that is it repetitive addition. Multiplication with transfinite ordinals doesn't always behave as we might expect from our general intuition developed from years of observation of the operation with finite values. We can give a formal definition of multiplication of ordinals as such:

$$\alpha \cdot \beta \triangleq \sum_{\gamma<\beta} \alpha.$$

So if we take the product of $4 \cdot 3$, for example, this means to add 4 three times, as such:

$$4 \cdot 3 = 4 + 4 + 4.$$

Now per the formal definition of addition we have

$$4 + 4 + 4 = \overline{4 \times \{0\} \cup 4 \times \{1\} \cup 4 \times \{2\}}$$

$$= \overline{\{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1), (0,2), (1,2), (2,2), (3,2)\}}.$$

This set is isomorphic to $\{0,1,2,3,4,5,6,7,8,9,10,11\} = 12$, and so $4 \cdot 3 = 12$.

To demonstrate multiplication with a transfinite ordinal it will now be shown that $\omega 2 \neq 2\omega$. In the first case $\omega 2$ means adding ω twice, and so is akin to $\omega + \omega$. As we saw in the section on addition $\omega + \omega > \omega$. A representation of this, to repeat from the section on addition, is given in Figure 5. The simpler notation is to represent this as $\omega 2$ rather than $\omega + \omega$.

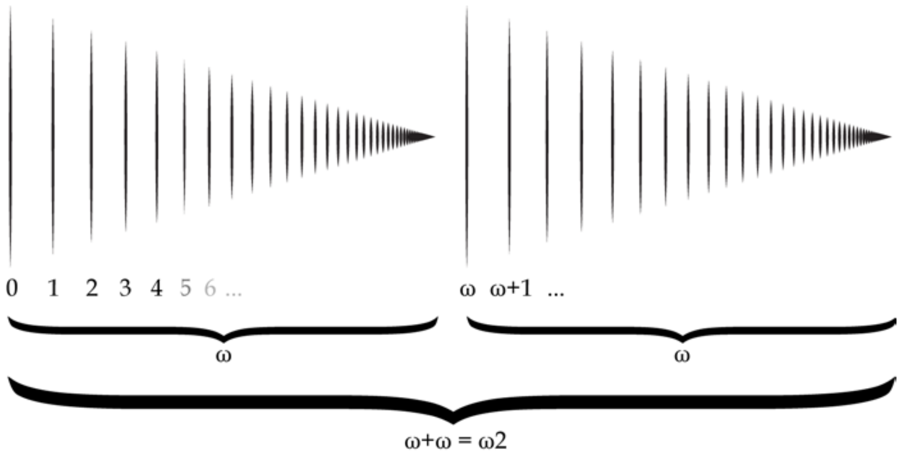

Figure 5. Visualization of $\omega 2$

As for 2ω, well this mean to add 2 endlessly. So $2\omega = 2 + 2 + 2 + \cdots$ which is further equal to $1 + 1 + 1 + \cdots$ which is equal to ω. Since $2\omega = \omega < \omega 2$ we can conclude that $2\omega \neq \omega 2$. So generally speaking, commutativity does not hold for ordinal multiplication, but we will show later some circumstances where it does hold.

Now addition per the previous section is cumbersome, and repetitive addition that is multiplication just makes the process even more troublesome. So, as with addition, we will provide a recursive definition of ordinal multiplication for more practical use. The following definition is adapted from Suppes' *Axiomatic Set Theory*.[26]

Recursive Definition of Ordinal Multiplication

1. $\alpha \cdot 0 \triangleq 0 \cdot \alpha \triangleq 0$

   Multiplying by zero on the left or right gives a product of zero.

2. $\alpha(\beta + 1) \triangleq \alpha\beta + \alpha$

[26] Suppes, Patrick. *Axiomatic Set Theory*. p 212.

Multiplying an ordinal α by a successor ordinal $\beta + 1$ is equal to $\alpha\beta + \alpha$. While this is consistent with our sense of a distributive law, further proof will be given that a distributive law exists in a more general sense.

3. If β is a limit ordinal, then

$$\alpha \cdot \beta \triangleq \bigcup_{\gamma<\beta} \alpha \cdot \gamma.$$

Assuming β is a limit ordinal, then multiplying by β on the right is equal to the supremum of all $\alpha\gamma$, where γ is every element of β. As was the case with the third part of the recursive definition of addition, this is regularly used in the third part of the transfinite induction process.

Results and Proofs

Theorem 2.0: $\boldsymbol{\alpha \neq 0 \wedge \beta \in K_{II} \rightarrow \alpha\beta \in K_{II}}$

Multiplying any ordinal, except for zero, on the right by a limit ordinal results in a limit ordinal. The reason for the exception is that multiplying any ordinal by zero on the left or the right results in zero, by the recursive definition of multiplication, which is not a limit ordinal.

Proof. [27,28] This will be proven by contradiction. We will first assume the premise of the hypothesis, that α is not zero and that β is a limit ordinal. As zero isn't a limit ordinal, we can also conclude that β is not zero. Therefore the product of α and β cannot be zero. Symbolically:

$$\alpha \neq 0 \wedge \beta \in K_{II} \rightarrow \alpha\beta \neq 0.$$

[27] Suppes, Patrick. *Axiomatic Set Theory*. p 214.
[28] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 64.

As every ordinal is a limit ordinal or not, we state that $\alpha\beta \in K_{II}$ or $\alpha\beta \in K_I$. We will assume that $\alpha\beta$ is not a limit ordinal, arrive at a contradiction, and then conclude that $\alpha\beta$ is a limit ordinal. If $\alpha\beta$ is a successor ordinal, then there is some γ that is its predecessor, such that:

$$(\exists\gamma)[\gamma + 1 = \alpha\beta].$$

As $\gamma < \gamma + 1$ and $\gamma + 1 = \alpha\beta$, certainly $\gamma < \alpha\beta$. We know from the definition of multiplication that

$$\alpha\beta = \bigcup_{\delta<\beta} \alpha\delta.$$

Given we know that $\gamma < \alpha\beta$, we can conclude that

$$\gamma < \bigcup_{\delta<\beta} \alpha\delta.$$

Per this conclusion there exists some $\delta_1 < \beta$ such that $\gamma < \alpha\delta_1$, which further implies that

$$\gamma + 1 < \alpha\delta_1 + 1.$$

As we know that $\alpha \neq 0$, which further means $\alpha \geq 1$, we can state

$$\alpha\delta_1 + 1 \leq \alpha\delta_1 + \alpha.$$

By the second part of the recursive definition of ordinal multiplication, we can further state that

$$\alpha\delta_1 + \alpha = \alpha(\delta_1 + 1).$$

Tying the last few statements together, we can further conclude that

$$\gamma + 1 < \alpha(\delta_1 + 1).$$

If β is a limit ordinal, and $\delta_1 < \beta$, then $\delta_1 + 1 < \beta$. Given $\gamma + 1 < \alpha(\delta_1 + 1)$ and $\delta_1 + 1 < \beta$ we conclude that $\gamma + 1 < \alpha\beta$ and

$$\gamma + 1 < \bigcup_{\delta<\beta} \alpha\delta = \alpha\beta = \gamma + 1.$$

As no ordinal can be a member of itself (or less than itself), we have thus reached our desired contradiction. We conclude then that $\alpha\beta \in K_{II}$.

■

<u>Theorem 2.1: $\boldsymbol{\alpha \neq 0 \wedge \beta < \gamma \leftrightarrow \alpha\beta < \alpha\gamma}$</u>

In other words α is not zero and $\beta < \gamma$ if and only if $\alpha\beta < \alpha\gamma$. The reason α cannot be zero is simply because if it were, then $\alpha\beta = \alpha\gamma$ because

$$\alpha\beta = 0 \cdot \beta = 0 = 0 \cdot \gamma = \alpha\gamma.$$

<u>Proof</u>.[29,30] This will be proven by induction on γ.

Part 1. If $\gamma = 0$, then our premise is false, as there is no $\beta < 0$. Thus the theorem holds vacuously.

Part 2. We will assume the hypothesis and show it holds for $\gamma + 1$. Per the hypothesis $\beta < \gamma$ and $\alpha\beta < \alpha\gamma$. Per our desire to see this hold for $\gamma + 1$, let's start with the fact that

$$\alpha(\gamma + 1) = \alpha\gamma + \alpha$$

by the recursive definition of ordinal addition.

Since $\alpha \neq 0$, it is clearly true that

$$\alpha\gamma < \alpha\gamma + \alpha.$$

By transitivity of the statement

$$\alpha\beta < \alpha\gamma < \alpha\gamma + \alpha = \alpha(\gamma + 1)$$

we have our desired result:

$$\alpha\beta < \alpha(\gamma + 1).$$

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29 Suppes, Patrick. *Axiomatic Set Theory*. p 213.
30 Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 63.

Part 3. γ is a limit ordinal: $\gamma \in K_{II}$. We will also assume the hypothesis holds for all elements of γ:

$$(\forall\delta)[\beta < \delta < \gamma \wedge \alpha \neq 0 \rightarrow \alpha\beta < \alpha\delta].$$

Further, as $\beta < \delta$, then certainly β is less than the supremum of δ:

$$\alpha\beta < \bigcup_{\delta<\gamma} \alpha\delta = \alpha\gamma,$$

and thus $\alpha\beta < \alpha\gamma$.

So we have shown $\alpha \neq 0 \wedge \beta < \gamma \rightarrow \alpha\beta < \alpha\gamma$. It will now be shown that the converse is also true, symbolically that $\alpha\beta < \alpha\gamma \rightarrow \alpha \neq 0 \wedge \beta < \gamma$.

If $\alpha\beta < \alpha\gamma$, then $\alpha \neq 0$, for the same reason as above. If $\beta = \gamma$, then $\alpha\beta = \alpha\gamma$, and vice versa. If $\gamma < \beta$ and $\alpha \neq 0$, then $\alpha\gamma < \alpha\beta$. Therefore $\alpha\beta < \alpha\gamma \rightarrow \beta < \gamma \wedge \alpha \neq 0$.

■

Theorem 2.2: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

This is our traditional distributive property.

Proof. [31,32] This will be proven by induction on γ.

Part 1. For the base case, we will assume $\gamma = 0$. Simply replacing γ with 0 and vice versa, we get

$$\alpha(\beta + \gamma) = \alpha(\beta + 0) = \alpha(\beta) = \alpha\beta = \alpha\beta + 0 = \alpha\beta + \alpha 0 = \alpha\beta + \alpha\gamma.$$

Part 2. We will assume the hypothesis and show that $\alpha\big(\beta + (\gamma + 1)\big) = \alpha\beta + \alpha(\gamma + 1)$.

Per the recursive definition of addition (alternatively by associativity of addition), we get

$$\alpha\big(\beta + (\gamma + 1)\big) = \alpha\big((\beta + \gamma) + 1\big).$$

[31] Suppes, Patrick. *Axiomatic Set Theory*. p 214.
[32] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 64.

Per the hypothesis

$$\alpha\big((\beta+\gamma)+1\big)=\alpha(\beta+\gamma)+\alpha=(\alpha\beta+\alpha\gamma)+\alpha.$$

Per associativity of addition

$$(\alpha\beta+\alpha\gamma)+\alpha=\alpha\beta+(\alpha\gamma+\alpha).$$

Finally, per the hypothesis

$$\alpha\beta+(\alpha\gamma+\alpha)=\alpha\beta+\alpha(\gamma+1).$$

Part 3. We will assume that γ is a limit ordinal: $\gamma\in K_{II}$ and also that the hypothesis holds for all elements of γ such that

$$\alpha(\beta+\delta)=\alpha\beta+\alpha\delta,\qquad \forall\delta<\gamma.$$

All that holding, there are two cases to consider: either $\alpha=0$ or $\alpha\neq 0$.

Case 1: $\alpha=0$.

Then by replacing α with 0 and vice versa:

$$\alpha(\beta+\gamma)=0(\beta+\gamma)=0=0+0=0\beta+0\gamma=\alpha\beta+\alpha\gamma.$$

Case 2: $\alpha\neq 0$.

Per Theorems 1.1 and 2.0, if γ is a limit ordinal, then so are $\beta+\gamma$ and $\alpha\gamma$. Thus by the recursive definitions of addition and multiplication

$$\alpha(\beta+\gamma)=\bigcup_{\delta<\beta+\gamma}\alpha\delta$$

and

$$\alpha\beta+\alpha\gamma=\bigcup_{\eta<\alpha\gamma}\alpha\beta+\eta.$$

If $\delta<\beta+\delta$ then

$$\delta<\beta\vee(\exists\tau)[\tau<\gamma\wedge\delta=\beta+\tau].$$

In the case where $\delta<\beta$, which implies $\delta<\beta$, by Theorem 2.1

$$\alpha\delta < \alpha\beta.$$

In the second case where $(\exists\tau)[\tau < \gamma \land \delta = \beta + \tau]$, then by the present clause, the distributive hypothesis, and with $\theta = \alpha\tau$

$$\alpha\delta = \alpha(\beta + \tau) = \alpha\beta + \alpha\tau = \alpha\beta + \theta.$$

Then since $\tau < \gamma$, then $\alpha\tau < \alpha\gamma$ and $\theta < \alpha\gamma$. Thus

$$\alpha(\beta + \gamma) \leq \alpha\beta + \alpha\gamma.$$

Now if $\eta < \alpha\gamma$ then

$$(\exists\delta)[\delta < \gamma \land \eta < \alpha\delta].$$

By Theorem 1.0 clearly

$$\beta + \delta < \beta + \gamma.$$

And further, since $\eta = \alpha\delta$, and by the distributive hypothesis

$$\alpha\beta + \eta < \alpha\beta + \alpha\delta = \alpha(\beta + \delta)$$

and

$$\alpha\beta + \alpha\gamma \leq \alpha(\beta + \gamma).$$

Therefore since $\alpha(\beta + \gamma) \leq \alpha\beta + \alpha\gamma \leq \alpha(\beta + \gamma)$, then

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$$

■

***Theorem 2.3:* $(\boldsymbol{\alpha\beta})\boldsymbol{\gamma} = \boldsymbol{\alpha}(\boldsymbol{\beta\gamma})$**

That is to say, ordinal multiplication is associative.

Proof. [33,34] This will be proven by induction on γ.

Part 1. For the base case, we will let $\gamma = 0$. By replacing γ with 0 and vice versa

[33] Suppes, Patrick. *Axiomatic Set Theory*. p 215.
[34] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 65.

$$(\alpha\beta)\gamma = (\alpha\beta)\cdot 0 = 0 = \alpha\cdot 0 = \alpha(\beta\cdot 0) = \alpha(\beta\gamma).$$

Part 2. We will assume the hypothesis and show that it holds for the successor; symbolically we will show that $(\alpha\beta)(\gamma+1) = \alpha\big(\beta(\gamma+1)\big)$.

By the distributive property

$$(\alpha\beta)(\gamma+1) = (\alpha\beta)\gamma + \alpha\beta.$$

By our hypothesis

$$(\alpha\beta)\gamma + \alpha\beta = \alpha(\beta\gamma) + \alpha\beta.$$

Then by the distributive property

$$\alpha(\beta\gamma) + \alpha\beta = \alpha(\beta\gamma + \beta) = \alpha\big(\beta(\gamma+1)\big).$$

Part 3. We will thus let γ a be limit ordinal, symbolically $\gamma \in K_{II}$. We will also assume the hypothesis holds for all elements of γ such that

$$(\alpha\beta)\delta = \alpha(\beta\delta), \quad \forall\delta < \gamma.$$

There are two cases here to consider: either $\alpha\beta = 0$ or $\alpha\beta \neq 0$.

Case 1: $\alpha\beta = 0$. If $\alpha\beta = 0$, then either $\alpha = 0$ or $\beta = 0$. Now if $\alpha\beta = 0$, then clearly

$$(\alpha\beta)\gamma = 0\cdot\gamma = 0.$$

Then since $\alpha = 0$ or $\beta = 0$, the latter of which implies $\beta\gamma = 0$, then

$$0 = a(\beta\gamma).$$

Case 2: $\alpha\beta \neq 0$. This implies that $\alpha \neq 0$ and $\beta \neq 0$. By definition of multiplying by a limit ordinal

$$(\alpha\beta)\gamma = \bigcup_{\delta<\gamma}(\alpha\beta)\delta.$$

Given that $\gamma \in K_{II}$, per Theorem 2.0 we also have $\beta\gamma \in K_{II}$ and

$$\alpha(\beta\gamma) = \bigcup_{\eta<\beta\gamma} \alpha\eta.$$

Now $\delta < \gamma$ implies by Theorem 2.1 that $\beta\delta < \beta\gamma$, and so

$$\bigcup_{\eta<\beta\gamma} \alpha\eta = \bigcup_{\beta\delta<\beta\gamma} \alpha(\beta\delta).$$

Therefore

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

■

Before we discuss the next conclusion, we will first prove that $2m = m2$, which will be used as the base case for Theorem 2.5.

***Lemma 2.4:* $\mathbf{2m = m2, \forall m < \omega}$**

Proof. This will be proven by induction on m. The three cases of $m = 0$, $m = 1$, and $m = 2$ are all trivial, so let us start with a base case where $m = 3$. So it will be shown that $2 \cdot 3 = 3 \cdot 2$. First start with the left hand side and we have

$$2 \cdot 3 = \overline{\{0,1\} \times \{0,1,2\}}$$

$$= \overline{\{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2)\}} = 6.$$

Now it will be shown that $3 \cdot 2 = 6$ as well:

$$3 \cdot 2 = \overline{\{0,1,2\} \times \{0,1\}}$$

$$= \overline{\{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\}} = 6.$$

Now for the induction, we will assume that $2m = m2$ and show this leads to $2(m+1) = (m+1) \cdot 2$. Let's start with the fact that

$$(m+1) \cdot 2 = (m+1) + (m+1).$$

As the addition of finite ordinals is both associative and commutative we have then

$$m + m + 1 + 1 = 2m + 2.$$

Lastly, by the distributive property, we have our desired conclusion:

$$2(m + 1) = (m + 1) \cdot 2.$$

■

Theorem 2.5: Multiplication of Finite Ordinals is Commutative

$$m \cdot n = n \cdot m, \qquad \forall m, n < \omega.$$

Similar to the commutativity of addition, this fact doesn't seem to get any attention. Again, theorems stated for ordinal arithmetic are generally reserved for results that apply to all ordinals. While there is no corollary like that for Theorem 1.4, and we won't use this theorem to prove anything, it is offered here for consideration to see that sometimes theorems that don't hold for absolutely all ordinals do hold under some strict circumstances. It's intuitively obvious, but difficult to prove the general case. It is easy to show this holds in any specific example like $3 \cdot 5 = 15 = 5 \cdot 3$. Here is a proof of the general case.

Proof. We will prove this by induction on n. As $n = 1$ and $n = 1$ are trivial cases, we will argue first the case where $n = 2$. By the previous proof, for any finite value of m:

$$m \cdot 2 = 2 \cdot m.$$

For the inductive step, we will assume that $m \cdot n = n \cdot m$ and show that $m(n + 1) = (n + 1)m$. We still start with

$$(n + 1)(m) = (n + 1)(1_1 + 1_2 + \cdots + 1_m).$$

Using the distributive property we have then

$$(n + 1)(1_1) + (n + 1)(1_2) + \cdots + (n + 1)(1_m)$$

$$= (n + 1)_1 + (n + 1)_2 + \cdots + (n + 1)_m$$

$$= (n_1 + 1_1) + (n_2 + 1_2) + \cdots + (n_m + 1_m).$$

Because the addition of finite ordinals is both associative and commutative, we have then

$$n_1 + n_2 + \cdots + n_m + 1_1 + 1_2 + \cdots + 1_m$$

$$= nm + m$$

$$= mn + m.$$

Finally by the distributive property, we have our desired result

$$(n + 1)m = m(n + 1).$$

■

Something that is useful in working out basic arithmetical operations is the knowledge of what happens when we multiply finite values by transfinite values. We will start with the base case of all these: what happens what we multiply by ω, the smallest transfinite ordinal.

***Theorem 2.6:* $\boldsymbol{m \cdot \omega = \omega,\ \forall m[0 < m < \omega]}$**

In other words, if we multiply any finite ordinal $m \neq 0$ by ω, the resulting product is equal to ω. The reason the finite ordinal m must not be 0, is because $0 \cdot \omega = 0$.

Proof. This will be proven constructively. We will start with a concrete example. For starters, the case where $m = 1$ is trivial, and is covered by the recursive definition of ordinal multiplication. So to cover a non-trivial case, let $m = 2$. Using the Cartesian product symbolism we have

$$2 \cdot \omega = \overline{\{0,1\} \times \{0,1,2,\dots\}} = \overline{\{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2),\dots\}} = \omega.$$

Per the nature of ordinal multiplication, and by the definition of the multiplicative identity

$$1 \cdot \omega = 1 + 1 + 1 + \cdots = \omega.$$

The same way we find that for any finite ordinal m, not zero,

$$m \cdot \omega = m + m + m + \cdots.$$

Given m is finite, we know that

$$m = 1_1 + 1_2 + 1_3 + \cdots + 1_m.$$

Therefore

$$m \cdot \omega = (1_1 + 1_2 + 1_3 + \cdots + 1_m) + (1_1 + 1_2 + 1_3 + \cdots + 1_m) + \cdots.$$

Since ordinal addition is associative by Theorem 1.2, we can drop the parenthesis and we have

$$m \cdot \omega = 1 + 1 + 1 + \cdots = \omega,$$

which is what we wanted to show.

■

Corollary 2.7: $\boldsymbol{m \cdot \alpha = \alpha,\ \forall m[0 < m < \omega],\ \forall \alpha \in K_{II}}$

In other words if we multiply any finite value by any limit ordinal α, the resulting product is α. Again, m can't be zero, or the theorem doesn't hold.

Proof. Any limit ordinal α can be expressed by the form $\alpha = \omega \cdot \beta$, so long as $\beta \neq 0$. So we have

$$m \cdot \alpha = m(\omega\beta).$$

Because ordinal multiplication is associative we have

$$m(\omega\beta) = (m\omega)\beta.$$

And finally, per Theorem 2.6 we know $m\omega = \omega$ and so

$$(m\omega)\beta = \omega\beta = \alpha.$$

And thus we have $m \cdot \alpha = \alpha$, as desired.

■

3. Ordinal Exponentiation

In the same way our understanding of multiplication is built up as repetitive addition, ordinal exponentiation will be formally defined as repetitive multiplication. Again, as was the case with the first two operations, while some things will behave similarly to our common notion of exponentiation, since we are dealing with numbers not addressed by our common notions of arithmetic, all such things will have to be proven.

Let us start with the more detailed definition of ordinal exponentiation, given as such

$$\alpha^{\beta} \triangleq \prod_{\gamma<\beta} \alpha.$$

Per this definition, we see that, for example

$$4^3 = \prod_{\gamma<3} 4 = 4_0 \cdot 4_1 \cdot 4_2$$

$$= (4+4+4+4) \cdot 4$$

$$= (16) \cdot 4 = 16 + 16 + 16 + 16 = 64.$$

This gets more interesting when we incorporate a transfinite ordinal. Let us look at the nature of ω^2. Squaring any ordinal simply means to multiply that ordinal twice, thus $\omega^2 = \omega \cdot \omega$. Since multiplying an ordinal by ω means to multiply it endlessly, we have further that

$$\omega^2 = \omega \cdot \omega = \omega + \omega + \omega + \cdots.$$

This statement can be further visualized by Figure 6.

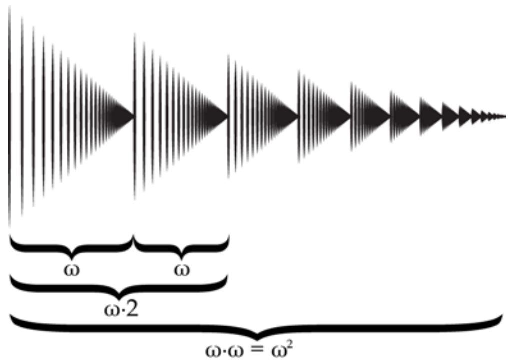

Figure 6. Visualization of ω^2

We can also take a value such as ω^2 and further multiply it. Figure 7 is a representation of $\omega^2 \cdot 8$. Any further attempt to carry this out to infinity will result in a virtually unintelligible image, due to a loss of fidelity.

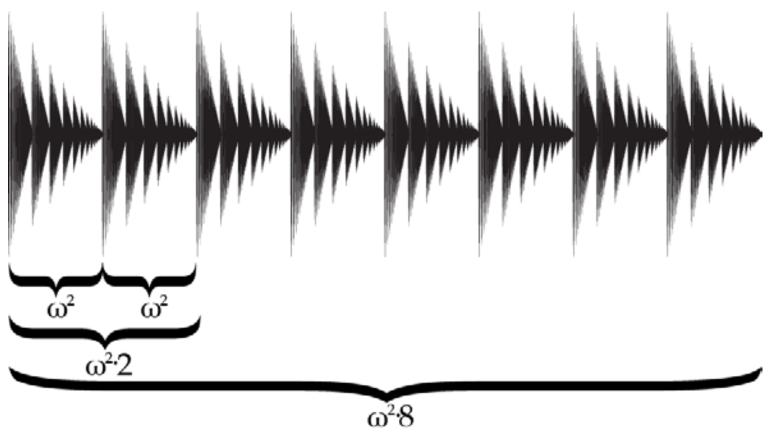

Figure 7. Visualization of Multiples of ω^2

As a single sum or a single product is already a messy bit of work, this is clearly going to get even worse when we try to compose this further based upon the standard of the Cartesian product. We will therefore carry this process no further, only to say that again we must multiply these in the order that they are written from left to right. We will then move forward with the more practical recursive definition of ordinal exponentiation, adapted from Suppes.[35]

Recursive Definition of Ordinal Exponentiation

1. $\alpha^0 \triangleq 1.$

   This holds true for any ordinal α, including zero. If zero is the exponent, the resulting power is always 1.

2. $\alpha^{\beta+1} \triangleq \alpha^\beta \cdot \alpha$

   If we take any ordinal α and raise it to a successor ordinal $\beta + 1$, then the result is the same as $\alpha^\beta \cdot \alpha$.

3. If β is a limit ordinal and $\alpha > 0$

$$\alpha^\beta \triangleq \bigcup_{\gamma<\beta} \alpha^\gamma.$$

   As with the recursive definitions of the first two arithmetical operations, this will be used regularly in the third part of the transfinite induction process. If $\alpha = 0$ while β is a limit ordinal, then $\alpha^\beta = 0^\beta = 0$.

Results and Proofs

Theorem 3.0: $\boldsymbol{\alpha > 1 \wedge \beta \in K_{II} \rightarrow \alpha^\beta \in K_{II}}$

For any ordinal $\alpha > 1$ and any limit ordinal β, α^β is also a limit ordinal.

Proof.[36] This will be proven by contradiction. Given that $\alpha > 1$ and $\beta \in K_{II}$, then also $\alpha^\beta > 1$. This also further implies that $\alpha^\beta \neq 0$. As is the case with any ordinal, $\alpha^\beta \in K_{II}$ and is of the class of limit ordinals, or it is not a limit ordinal and $\alpha^\beta \in K_I$. We will assume the

[35] Suppes, Patrick. *Axiomatic Set Theory*. p 215.
[36] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 69.

latter case, that α^β is not a limit ordinal, which further implies that α^β has an immediate predecessor δ such that $\delta + 1 = \alpha^\beta$.

Per the recursive definition of ordinal exponentiation, given $\alpha \neq 0$ and $\beta \in K_{II}$

$$\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma.$$

Given $\delta < \delta + 1$ and $\delta + 1 = \alpha^\beta$, we derive that $\delta < \alpha^\beta$. So then there exists some $\gamma < \beta$ such that $\delta < \alpha^\gamma$. Since $\alpha > 1$, and also because $\delta + 1 = \alpha^\beta$,

$$\delta + 1 \leq \alpha^\gamma < \alpha^{\gamma+1}.$$

But since β is a limit ordinal, $\gamma < \beta$ implies $\gamma + 1 < \beta$ and so

$$\delta + 1 < \alpha^\beta = \delta + 1,$$

Which is our desired contradiction. Therefore α^β is a limit ordinal.

■

Theorem 3.1: $\alpha^\beta \alpha^\gamma = \alpha^{\beta+\gamma}$

In other words, when we multiply exponential expression with the same base, we simply add the exponents. This is consistent with our general intuition about how exponents behave. However, we cannot assume that our intuition about finite numbers translates to good behavior of our transfinite ordinal numbers.

Proof. [37] This will be proven by our transfinite induction on γ.

Part 1. For the base case, we will let $\gamma = 0$. Then by simple substitution of γ and 0

$$\alpha^\beta \alpha^\gamma = \alpha^\beta \alpha^0 = \alpha^\beta \cdot 1 = \alpha^\beta = \alpha^{\beta+0} = \alpha^{\beta+\gamma}.$$

Part 2. We will assume the hypothesis and show it leads to $\alpha^\beta \alpha^{\gamma+1} = \alpha^{\beta+\gamma+1}$. By the recursive definition of ordinal exponentiation

[37] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 69.

$$\alpha^\beta \alpha^{\gamma+1} = \alpha^\beta \alpha^\gamma \alpha.$$

By our hypothesis

$$\alpha^\beta \alpha^\gamma \alpha = \alpha^{\beta+\gamma} \alpha.$$

Once again by the definition of ordinal exponentiation we have our desired result

$$\alpha^{\beta+\gamma} \alpha = \alpha^{\beta+\gamma+1}.$$

Part 3. We will now let γ be a limit ordinal and assume the hypothesis holds for all elements of γ:

$$\alpha^\beta \alpha^\delta = \alpha^{\beta+\delta} \;\; \forall \delta < \gamma.$$

We will now consider three cases where $\alpha = 0$, another where $\alpha = 1$, and finally the case in which $\alpha > 1$. First, if $\alpha = 0$, then $\alpha^\gamma = 0$ and $\alpha^{\beta+\gamma} = 0$, and thus

$$\alpha^\beta \alpha^\gamma = 0 = \alpha^{\beta+\gamma}.$$

If $\alpha = 1$, then by substitution

$$\alpha^\beta \alpha^\gamma = 1^\beta \cdot 1^\gamma = 1 = 1^{\beta+\gamma} = \alpha^{\beta+\gamma}.$$

If $\alpha > 1$, then by Theorem 3.0 and our given assumption that γ is a limit ordinal, α^γ is also a limit ordinal. So by our definition of multiplying by a limit ordinal

$$\alpha^\beta \alpha^\gamma = \bigcup_{\delta < \alpha^\gamma} \alpha^\beta \delta.$$

Because $\delta < \alpha^\gamma$ there exists some $\tau < \gamma$ such that $\delta < \alpha^\tau$. By our hypothesis $\tau < \gamma$ implies

$$\alpha^\beta \alpha^\tau = \alpha^{\beta+\tau}.$$

By Theorem 1.0

$$\beta + \tau < \beta + \gamma.$$

Now $\delta < \alpha^\tau$ implies $\alpha^\beta \delta \leq \alpha^\beta \alpha^\tau$. Per our hypothesis $\alpha^\beta \alpha^\tau = \alpha^{\beta+\tau}$. This leads to $\alpha^\beta \delta \leq \alpha^{\beta+\tau}$, which leads us to

$$\alpha^{\beta}\alpha^{\gamma} \leq \alpha^{\beta+\gamma}.$$

Next, given γ is a limit ordinal, by Theorem 1.1 $\beta + \gamma$ is also a limit ordinal. Then by definition of exponentiation by a limit ordinal

$$\alpha^{\beta+\gamma} = \bigcup_{\eta<\beta+\gamma} \alpha^{\eta}.$$

If $\eta < \beta + \gamma$ then either $\eta \leq \beta$ or $\eta > \beta$, the latter of which implies the existence of some τ such that

$\eta = \beta + \tau$. In the first case, if $\eta \leq \beta$ then $\alpha^{\eta} \leq \alpha^{\beta} \cdot 1$, and because $\alpha > 1$ we also know that $\alpha^{\eta} > 1$. Otherwise if $\eta = \beta + \tau$ then $\tau < \gamma$. So by our hypothesis

$$\alpha^{\beta+\tau} = \alpha^{\beta}\alpha^{\tau}.$$

Further $\alpha^{\eta} = \alpha^{\beta+\tau} = \alpha^{\beta}\alpha^{\tau}$ and $\alpha^{\tau} < \alpha^{\gamma}$. Therefore

$$\alpha^{\beta}\alpha^{\gamma} = \alpha^{\beta+\gamma}.$$

■

***Theorem 3.2:* $\alpha < \beta \wedge \gamma > 1 \rightarrow \gamma^{\alpha} < \gamma^{\beta}$**

In other words, exponentiation preserves order.

Proof.[38] This will be proven by induction on β.

Part 1. The base case is that β is the immediate successor of α, such that $\beta = \alpha + 1$. Now since

$\gamma > 1$ we have

$$\gamma^{\alpha} < \gamma^{\alpha}\gamma.$$

By Theorem 3.1 we have

$$\gamma^{\alpha}\gamma = \gamma^{\alpha+1}.$$

[38] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 67.

Finally, since $\beta = \alpha + 1$ we have

$$\gamma^{\alpha+1} = \gamma^{\beta}.$$

Thus $\gamma^{\alpha} < \gamma^{\beta}$.

Part 2. Assume the hypothesis and show that $\alpha < \beta + 1$ yields $\gamma^{\alpha} < \gamma^{\beta+1}$. If $\alpha < \beta + 1$ then $\alpha \leq \beta$. Thus

$$\gamma^{\alpha} \leq \gamma^{\beta}.$$

Further, as $\gamma > 1$ we have $\gamma^{\beta} < \gamma^{\beta}\gamma = \gamma^{\beta+1}$. Thus

$$\gamma^{\alpha} < \gamma^{\beta+1}.$$

Part 3. Let β be a limit ordinal. By definition of exponentiation by a limit ordinal

$$\gamma^{\beta} = \bigcup_{\delta<\beta} \gamma^{\delta}.$$

Since $\beta \in K_{II}$, if $\alpha < \beta$, then $\alpha + 1 < \beta$. So

$$\gamma^{\alpha} < \bigcup_{\delta<\beta} \gamma^{\delta} = \gamma^{\beta}.$$

■

***Theorem 3.3:* $(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}$**

In other words, if take the power of α^{β} and exponentiate it by γ, the result is the same as exponentiating α by the product of $\beta\gamma$. This aligns with our ordinary sense of exponents.

Proof.[39] This will be proven by induction on γ.

Part 1. The base case is where $\gamma = 0$. By simple substitution between γ and 0:

$$(\alpha^{\beta})^{\gamma} = (\alpha^{\beta})^{0} = 1 = \alpha^{0} = \alpha^{\beta\cdot 0} = \alpha^{\beta\gamma}.$$

Part 2. Assume the hypothesis and show that $(\alpha^{\beta})^{\gamma+1} = \alpha^{\beta(\gamma+1)}$.

[39] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 70.

By the recursive definition of ordinal exponentiation

$$\left(\alpha^{\beta}\right)^{\gamma+1} = \left(\alpha^{\beta}\right)^{\gamma}\alpha^{\beta}.$$

By our hypothesis $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{\beta\gamma}$ and so

$$\left(\alpha^{\beta}\right)^{\gamma}\alpha^{\beta} = \alpha^{\beta\gamma}\alpha^{\beta}.$$

By Theorem 3.1 we have then

$$\alpha^{\beta\gamma}\alpha^{\beta} = \alpha^{\beta\gamma+\beta}.$$

Finally, using the distributive property we have

$$\alpha^{\beta\gamma+\beta} = \alpha^{\beta(\gamma+1)}.$$

Part 3. Let γ be a limit ordinal and let the hypothesis hold for all elements of γ such that

$$\left(\alpha^{\beta}\right)^{\delta} = \alpha^{\beta\delta} \;\; \forall\delta < \gamma.$$

We will then do this by two cases: either $\beta = 0$ or $\beta \neq 0$.

In the first case, where $\beta = 0$, by simple substitution we find the desired equality as such:

$$\left(\alpha^{\beta}\right)^{\gamma} = (\alpha^{0})^{\gamma} = 1^{\gamma} = 1 = \alpha^{0} = \alpha^{0\cdot\gamma} = \alpha^{\beta\gamma}.$$

In the second case, where $\beta \neq 0$, we must consider two subcases: either $\alpha = 0$ or $\alpha \neq 0$. In the first case, where $\alpha = 0$ then by substitution we have

$$\left(\alpha^{\beta}\right)^{\gamma} = \left(0^{\beta}\right)^{\gamma} = 0 = 0^{\beta\gamma} = \alpha^{\beta\gamma}.$$

In the case where $\alpha \neq 0$, then also we have $\alpha^{\beta} \neq 0$. As γ is a limit ordinal, by the recursive definition of ordinal exponentiation we have

$$\left(\alpha^{\beta}\right)^{\gamma} = \bigcup_{\delta<\gamma}\left(\alpha^{\beta}\right)^{\delta}.$$

Given $\delta < \gamma$, we have by our hypothesis

$$\left(\alpha^{\beta}\right)^{\delta} = \alpha^{\beta\delta}.$$

Also, given $\delta < \gamma$, by Theorem 2.1 we have

$$\beta\delta < \beta\gamma$$

and

$$\left(\alpha^{\beta}\right)^{\gamma} \leq \alpha^{\beta\gamma}.$$

As $\gamma \in K_{II}$, by Theorem 2.0 we have also $\beta\gamma \in K_{II}$, and by the definition of ordinal exponentiation

$$\alpha^{\beta\gamma} = \bigcup_{\eta<\beta\gamma} \alpha^{\eta}$$

Given $\eta < \beta\gamma$ there exists some $\delta < \gamma$ such that $\eta < \beta\delta$. By Theorem 3.2 we have $\alpha^{\eta} < \alpha^{\beta\delta}$ and so

$$\alpha^{\eta} \leq \alpha^{\beta\delta}.$$

Therefore $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{\beta\gamma}$.

■

<u>Theorem 3.4: $\alpha < \beta \to \alpha^{\gamma} \leq \beta^{\gamma}$</u>

<u>Proof</u>.[40] This will be proven by transfinite induction on γ.

Part 1. For the base case, we will let $\gamma = 0$, and we will see that $\alpha^{\gamma} = \alpha^{\beta}$ by substitution:

$$\alpha^{\gamma} = \alpha^{0} = 1 = \beta^{0} = \beta^{\gamma}.$$

Part 2. We will assume the hypothesis and show that $\alpha^{\gamma+1} < \beta^{\gamma+1}$. By the recursive definition of ordinal exponentiation we have

$$\alpha^{\gamma+1} = \alpha^{\gamma}\alpha.$$

[40] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* p 68.

By the hypothesis, monotonicity of multiplication, and the recursive definition of ordinal exponentiation we have

$$\alpha^{\gamma+1} = \alpha^{\gamma}\alpha \leq \beta^{\gamma}\alpha < \beta^{\gamma}\beta = \beta^{\gamma+1}.$$

Part 3. We will assume γ is a limit ordinal, and then show the hypothesis still holds. Firstly, by definition of having a limit ordinal as an exponent we have

$$\alpha^{\gamma} = \bigcup_{\delta<\gamma} \alpha^{\delta}.$$

We know that $\alpha^{\delta} \leq \beta^{\delta}$, and therefore

$$\bigcup_{\delta<\gamma} \alpha^{\delta} \leq \bigcup_{\delta<\gamma} \beta^{\delta}.$$

By recursive definition of ordinal exponentiation

$$\bigcup_{\delta<\gamma} \beta^{\delta} = \beta^{\gamma}.$$

Thus we have our desired result

$$\alpha^{\gamma} \leq \beta^{\gamma}.$$

■

Theorem 3.5

If $\alpha \in K_{II} \wedge \beta > 0$ and $0 < m < \omega$ then a recursive definition of the statement $\left(\alpha^{\beta}m\right)^{\gamma}$ is as follows

1. $\left(\alpha^{\beta}m\right)^{\gamma} = 1$, when $\gamma = 0$
2. $\left(\alpha^{\beta}m\right)^{\gamma} = \alpha^{\beta\gamma}m$, when $\gamma \in K_I$ and $\gamma \neq 0$
3. $\left(\alpha^{\beta}m\right)^{\gamma} = \alpha^{\beta\gamma}$, when $\gamma \in K_{II}$.

Proof.[41] This will be proven by induction on γ. Further, each portion of the transfinite induction proves each corresponding portion of the recursive definition of the theorem.

Part 1. The case where $\gamma = 0$ is trivial. By simple substitution we have

$$(\alpha^{\beta} m)^{\gamma} = (\alpha^{\beta} m)^{0} = 1.$$

Part 2. Assume the hypothesis and show then that $(\alpha^{\beta} m)^{\gamma+1} = \alpha^{\beta(\gamma+1)} m$.

If we consider the base case for the successor ordinals as $\gamma = 1$, then we have by substitution

$$(\alpha^{\beta} m)^{\gamma} = (\alpha^{\beta} m)^{1} = \alpha^{\beta} m = \alpha^{\beta \cdot 1} m = \alpha^{\beta\gamma} m,$$

and we see the definition holds. Let us then consider the general successor ordinal. By our recursive definition of ordinal exponentiation we have

$$(\alpha^{\beta} m)^{\gamma+1} = (\alpha^{\beta} m)^{\gamma} \alpha^{\beta} m.$$

By our hypothesis we have

$$(\alpha^{\beta} m)^{\gamma} \alpha^{\beta} m = \alpha^{\beta\gamma} m \cdot \alpha^{\beta} m.$$

As $\alpha \in K_{II}$ and $\beta \neq 0$, we have also $\alpha^{\beta} \in K_{II}$. By Corollary 2.7, since m is a finite ordinal we have also that

$$m \cdot \alpha^{\beta} = \alpha^{\beta},$$

and so

$$\alpha^{\beta\gamma} m \cdot \alpha^{\beta} m = \alpha^{\beta\gamma} \alpha^{\beta} m.$$

By Theorem 3.1 we have

$$\alpha^{\beta\gamma} \alpha^{\beta} m = \alpha^{\beta\gamma+\beta} m.$$

Finally, by the distributive property we have

$$\alpha^{\beta\gamma+\beta} m = \alpha^{\beta(\gamma+1)} m.$$

[41] Takeuti, Gaisi. *Introduction to Axiomatic Set Theory.* pp 71-72.

Part 3. Let γ be a limit ordinal and assume the hypothesis holds for all elements of γ as such:

$$\left(\alpha^{\beta}m\right)^{\delta} = \alpha^{\beta\delta}m, \qquad \forall\delta < \gamma.$$

Per theorem 3.4, we have

$$\left(\alpha^{\beta}\right)^{\gamma} \leq \left(\alpha^{\beta}m\right)^{\gamma}.$$

By definition of using a limit ordinal as an exponent we have

$$\left(\alpha^{\beta}m\right)^{\gamma} = \bigcup_{\delta<\gamma}\left(\alpha^{\beta}m\right)^{\delta}.$$

Per our hypothesis $\left(\alpha^{\beta}m\right)^{\delta} = a^{\beta\delta}m$. In the union set of these two, we lose strict equality as the two are only necessarily equal when $m = 1$:

$$\bigcup_{\delta<\gamma}\left(\alpha^{\beta}m\right)^{\delta} \leq \bigcup_{\delta<\gamma}\alpha^{\beta\delta}m.$$

As $m < \alpha^{\beta}$, we have also that $\alpha^{\beta\delta}m < \alpha^{\beta\delta}\alpha^{\beta}$, per Theorem 2.1. Further, by the distributive property (Theorem 2.2) and by the definition of ordinal exponentiation we also know that $\alpha^{\beta\delta}\alpha^{\beta} = \alpha^{\beta\delta+\beta} = \alpha^{\beta(\delta+1)}$. Therefore we also have $\alpha^{\beta\delta}m < \alpha^{\beta(\delta+1)}$. Taking the union set of both of these for all $\delta < \gamma$, we lose strict inequality and have thus

$$\bigcup_{\delta<\gamma}\alpha^{\beta\delta}m \leq \bigcup_{\delta<\gamma}\alpha^{\beta(\delta+1)}.$$

Since γ is a limit ordinal, if $\delta < \gamma$ then also $\delta + 1 < \gamma$ we have

$$\bigcup_{\delta<\gamma}\alpha^{\beta(\delta+1)} = \bigcup_{\delta<\gamma}\alpha^{\beta\delta}.$$

Then by the definition of ordinal exponentiation we have

$$\bigcup_{\delta<\gamma}\alpha^{\beta\delta} = \alpha^{\beta\gamma}.$$

Thus

$$\left(\alpha^{\beta}m\right)^{\gamma} \leq \alpha^{\beta\gamma}.$$

Combined with the previous conclusion of $\left(\alpha^{\beta}\right)^{\gamma} \leq \left(\alpha^{\beta}m\right)^{\gamma}$ we have

$$\left(\alpha^{\beta}\right)^{\gamma} \leq \left(\alpha^{\beta}m\right)^{\gamma} \leq \alpha^{\beta\gamma}.$$

Since by Theorem 3.3 we have $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{\beta\gamma}$, we finally conclude that

$$\left(\alpha^{\beta}m\right)^{\gamma} = \alpha^{\beta\gamma}.$$

■

For example, it can be shown that $(\omega 2)^{\omega} = \omega^{\omega}$. This is by no means a trivial solution, nor directly intuitive without the given proof. We see that $(\omega 2)^{\omega} = \omega 2 \cdot \omega 2 \cdot \omega 2 \ldots$, and by the associative property of multiplication is also equal to $\omega \cdot 2\omega \cdot 2\omega \cdot 2\omega \ldots$ It is not immediately clear whether this product would end with a 2 or end with an ω, but in actuality there is no "last" term. So without this proven Theorem 3.5, we might be stuck forever in wondering if $(\omega 2)^{\omega} = \omega^{\omega}$, as is stated will be the actual result, or if $(\omega 2)^{\omega} = \omega^{\omega}2$. Per Theorem 3.5, we thus have

$$\omega^{\omega} \leq (\omega 2)^{\omega} = \bigcup_{\alpha<\omega} (\omega 2)^{\alpha} = \bigcup_{\alpha<\omega} \omega^{\alpha}2 \leq \bigcup_{\alpha<\omega} \omega^{\alpha+1} = \omega^{\omega}$$

and by the same sort of sandwich, we have $(\omega 2)^{\omega} = \omega^{\omega}$.

We will conclude this section on ordinal exponentiation with an important result that simplifies many statements in the arithmetical tables at the end of the paper. This is similar in form to that of Theorems 1.6 and 2.6. There is not, however, a more generalized form of this theorem, as there was with Corollaries 1.7 and 2.7.

<u>*Theorem 3.6:* $m^{\omega} = \omega,\ \forall m[1 < m < \omega]$</u>

If we raise any finite ordinal greater than 1 to the exponent of ω, the resulting power is always ω. As for any exponent on 1 or 0 produces 1 or 0, respectively, those bases don't hold for this theorem.

<u>Proof</u>. This will be proven constructively. Let us first start by looking at the meaning of our premise:

$$m^{\omega} = m \cdot m \cdot m \cdot \cdots.$$

Now let us look at a series of progressively higher exponents, starting with 1 (note that the ellipses in the following statements do not imply counting infinitely, as each of the following expressions representing m^n, where $n < \omega$, is finite. For starters

$$m^1 = (1_1 + 1_2 + \cdots + 1_m).$$

Now if we progress to m^2, this just means to multiply the m^1 term to itself:

$$m^2 = m \cdot m = (1_1 + 1_2 + \cdots + 1_m) \cdot (1_1 + 1_2 + \cdots + 1_m).$$

By the distributive property, we then have

$$(1_1 + 1_2 + \cdots + 1_m)(1_1) + (1_1 + 1_2 + \cdots + 1_m)(1_2) + \cdots + (1_1 + 1_2 + \cdots + 1_m)(1_m).$$

By our recursive definition of ordinal multiplication we know that $\alpha \cdot 1 = \alpha$ for any ordinal α, and so then we have

$$(1_1 + 1_2 + \cdots + 1_m)_1 + (1_1 + 1_2 + \cdots + 1_m)_2 + \cdots + (1_1 + 1_2 + \cdots + 1_m)_m.$$

Which means the number of times we counted to m was m. In a form of conveniently shorter length, we could further write this as

$$1_1 + 1_2 + \cdots + 1_{m^2}.$$

As one last concrete example, let's progress to m^3:

$$m^3 = m^2 \cdot m = (1_1 + 1_2 + \cdots + 1_{m^2}) \cdot (1_1 + 1_2 + \cdots + 1_m).$$

By the same rationale as above, we have then

$$(1_1 + 1_2 + \cdots + 1_{m^2})(1_1) + (1_1 + 1_2 + \cdots + 1_{m^2})(1_2) + \cdots + (1_1 + 1_2 + \cdots + 1_{m^2})(1_m)$$

$$= (1_1 + 1_2 + \cdots + 1_{m^2})_1 + (1_1 + 1_2 + \cdots + 1_{m^2})_2 + \cdots + (1_1 + 1_2 + \cdots + 1_{m^2})_m$$

$$= 1_1 + 1_2 + \cdots + 1_{m^3}.$$

With each successive case, we are simply adding a finite number to the previous amount, a finite sequence of ones. Therefore the union set

$$m^\omega = \bigcup_{n<\omega} m^n = 1 + 1 + 1 + \cdots = \omega.$$

■

4. Conclusion

The study of ordinals exists as a foundation for the study of cardinal numbers. As Keith Devlin puts it, "the cardinality of X, denoted by $|X|$, is the least ordinal α for which there exists a bijection $f: \alpha \leftrightarrow X$." He goes on further to prove that every cardinal number is a limit ordinal. What follows here is an adaptation of said proof, by way of demonstrating that any successor ordinal, which by its nature is not a limit ordinal, can be put in a one-to-one relationship with a smaller ordinal, and therefore is not a cardinal.[42]

Theorem 4.0: Every Transfinite Cardinal is a Limit Ordinal

Proof.[43] Let α be a transfinite ordinal, so $\alpha \geq \omega$. It will then be shown that $\alpha + 1$ is not a cardinal. Define a recursive mapping $f: \alpha \rightarrow \alpha + 1$ as such:

1. $f(0) = \alpha$
2. $f(n+1) = n,$ for $n < \omega$
3. $f(\beta) = \beta$, if $\omega \leq \beta < \alpha$.

Per this recursive formula, we see that f is a bijection. Therefore $\alpha + 1$ is not a cardinal.

∎

For example, let $\alpha = \omega$, and then show $\alpha + 1 = \omega + 1$ is not a cardinal. Recall the set of elements of $\omega = \{0,1,2, \dots\}$ and $\omega + 1 = \{0,1,2, \dots, \omega\}$. The mapping for $f: \alpha \rightarrow \alpha + 1$ would be layed out as follows.

$x \in \omega$	0	1	2	3	…
$f(x) \in \omega + 1$	ω	0	1	2	…

[42] Devlin, Keith. *The Joy of Sets*. p 76.
[43] Ibid, p 76.

We can see that the two ordinals have the same cardinality and thus we have a bijection.

As another example let $\alpha = \omega 2 + 3$, then $\alpha + 1 = \omega 2 + 4$. The mapping for f would then be laid out per the following table, from which we once again see the resulting bijection.

$x \in \omega 2 + 3$	0	1	2	3	…	ω	$\omega + 1$	$\omega + 2$	…	$\omega 2$	$\omega 2 + 1$	$\omega 2 + 2$
$f(x) \in \omega 3 + 4$	$\omega 2 + 3$	0	1	2	…	ω	$\omega + 1$	$\omega + 2$	…	$\omega 2$	$\omega 2 + 1$	$\omega 2 + 2$

The converse is not true; not every limit ordinal is a cardinal. To show this, let us show that the limit ordinal $\omega 2$ can be put in a one-to-one relationship with ω. Recall that $\omega 2 = \omega + \omega = \{0,1,2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$. The following table shows a mapping that demonstrates a bijection. Essentially, the second half of $\omega 2$ is zipped up in alternating terms with the first half.

$x \in \omega$	0	1	2	3	4	5	6	7	8	…
$f(x) \in \omega 2$	ω	0	$\omega + 1$	1	$\omega + 2$	2	$\omega + 3$	3	$\omega + 4$	…

In fact, while it won't be shown here, all arithmetical operations on ω, with or without finite values, are said to have cardinality of $\aleph_0$, is the cardinality of the set of all natural numbers. Since ω is the smallest ordinal with which said ordinals can be put into a one-to-one relationship, per the given definition of cardinality above we can conclude that $\aleph_0 = \omega$. Because of this equality, ω is often denoted ω_0. The supremum of all arithmetical operation of all ordinals that have a cardinality of $\aleph_0$ is called ω_1, and $\omega_1 = \aleph_1$. If Cantor's Continuum Hypothesis is true, then $\aleph_1$ is the cardinality of the real numbers.

What has been presented here is enough to build up a sense of arithmetical operations of ordinal numbers. In conclusion, here is a series of tables that outline many results of the three arithmetical operations discussed here, limited to ordinal values less than ω_1 for brevity. Much of the journey of this research has been motivated by the completion of these tables of results. They have been a critical part of the work at the beginning, middle, and the end result of this paper.

Table of Ordinal Addition

$+$	0	1	2	⋯	ω	$\omega 2$	$\omega 3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
0	0	1	2	⋯	ω	$\omega 2$	$\omega 3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
1	1	2	3	⋯	ω	$\omega 2$	$\omega 3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
2	2	3	4	⋯	ω	$\omega 2$	$\omega 3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω	ω	$\omega + 1$	$\omega + 2$	⋯	$\omega 2$	$\omega 3$	$\omega 4$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\omega 2$	$\omega 2$	$\omega 2 + 1$	$\omega 2 + 2$	⋯	$\omega 3$	$\omega 4$	$\omega 5$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\omega 3$	$\omega 3$	$\omega 3 + 1$	$\omega 3 + 2$	⋯	$\omega 4$	$\omega 5$	$\omega 6$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^2	ω^2	$\omega^2 + 1$	$\omega^2 + 2$	⋯	$\omega^2 + \omega$	$\omega^2 + \omega 2$	$\omega^2 + \omega 3$	⋯	$\omega^2 2$	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
ω^3	ω^3	$\omega^3 + 1$	$\omega^3 + 2$	⋯	$\omega^3 + \omega$	$\omega^3 + \omega 2$	$\omega^3 + \omega 3$	⋯	$\omega^3 + \omega^2$	$\omega^3 2$	ω^4	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
ω^4	ω^4	$\omega^4 + 1$	$\omega^4 + 2$	⋯	$\omega^4 + \omega$	$\omega^4 + \omega 2$	$\omega^4 + \omega 3$	⋯	$\omega^4 + \omega^2$	$\omega^4 + \omega^3$	$\omega^4 2$	⋯	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^ω	ω^ω	$\omega^\omega + 1$	$\omega^\omega + 2$	⋯	$\omega^\omega + \omega$	$\omega^\omega + \omega 2$	$\omega^\omega + \omega 3$	⋯	$\omega^\omega + \omega^2$	$\omega^\omega + \omega^3$	$\omega^\omega + \omega^4$	⋯	$\omega^\omega 2$	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\omega^{\omega 2}$	$\omega^{\omega 2}$	$\omega^{\omega 2} + 1$	$\omega^{\omega 2} + 2$	⋯	$\omega^{\omega 2} + \omega$	$\omega^{\omega 2} + \omega 2$	$\omega^{\omega 2} + \omega 3$	⋯	$\omega^{\omega 2} + \omega^2$	$\omega^{\omega 2} + \omega^3$	$\omega^{\omega 2} + \omega^4$	⋯	$\omega^{\omega 2} + \omega^\omega$	$\omega^{\omega 2} 2$	$\omega^{\omega 3}$
$\omega^{\omega 3}$	$\omega^{\omega 3}$	$\omega^{\omega 3} + 1$	$\omega^{\omega 3} + 2$	⋯	$\omega^{\omega 3} + \omega$	$\omega^{\omega 3} + \omega 2$	$\omega^{\omega 3} + \omega 3$	⋯	$\omega^{\omega 3} + \omega^2$	$\omega^{\omega 3} + \omega^3$	$\omega^{3\omega} + \omega^4$	⋯	$\omega^{\omega 3} + \omega^\omega$	$\omega^{\omega 3} + \omega^{\omega 2}$	$\omega^{\omega 3} 2$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^{ω^ω}	ω^{ω^ω}	$\omega^{\omega^\omega} + 1$	$\omega^{\omega^\omega} + 2$	⋯	$\omega^{\omega^\omega} + \omega$	$\omega^{\omega^\omega} + \omega 2$	$\omega^{\omega^\omega} + \omega 3$	⋯	$\omega^{\omega^\omega} + \omega^2$	$\omega^{\omega^\omega} + \omega^3$	$\omega^{\omega^\omega} + \omega^4$	⋯	$\omega^{\omega^\omega} + \omega^\omega$	$\omega^{\omega^\omega} + \omega^{\omega 2}$	$\omega^{\omega^\omega} + \omega^{\omega 3}$
$\omega^{\omega^{\omega 2}}$	$\omega^{\omega^{\omega 2}}$	$\omega^{\omega^{\omega 2}} + 1$	$\omega^{\omega^{\omega 2}} + 2$	⋯	$\omega^{\omega^{\omega 2}} + \omega$	$\omega^{\omega^{\omega 2}} + \omega 2$	$\omega^{\omega^{\omega 2}} + \omega 3$	⋯	$\omega^{\omega^{\omega 2}} + \omega^2$	$\omega^{\omega^{\omega 2}} + \omega^3$	$\omega^{\omega^{\omega 2}} + \omega^4$	⋯	$\omega^{\omega^{\omega 2}} + \omega^\omega$	$\omega^{\omega^{\omega 2}} + \omega^{\omega 2}$	$\omega^{\omega^{\omega 2}} + \omega^{\omega 3}$
$\omega^{\omega^{\omega 3}}$	$\omega^{\omega^{\omega 3}}$	$\omega^{\omega^{\omega 3}} + 1$	$\omega^{\omega^{\omega 3}} + 2$	⋯	$\omega^{\omega^{\omega 3}} + \omega$	$\omega^{\omega^{\omega 3}} + \omega 2$	$\omega^{\omega^{\omega 3}} + \omega 3$	⋯	$\omega^{\omega^{\omega 3}} + \omega^2$	$\omega^{\omega^{\omega 3}} + \omega^3$	$\omega^{\omega^{\omega 3}} + \omega^4$	⋯	$\omega^{\omega^{\omega 3}} + \omega^\omega$	$\omega^{\omega^{\omega 3}} + \omega^{\omega 2}$	$\omega^{\omega^{\omega 3}} + \omega^{\omega 3}$

Table of Ordinal Multiplication

$\cdot$	**0**	**1**	**2**	$\cdots$	ω	$\omega 2$	$\omega 3$	$\cdots$	ω^2	ω^3	ω^4	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
0	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0	$\cdots$	0	0	0
1	0	1	2	$\cdots$	ω	$\omega 2$	$\omega 3$	$\cdots$	ω^2	ω^3	ω^4	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
2	0	3	4	$\cdots$	ω	$\omega 2$	$\omega 3$	$\cdots$	ω^2	ω^3	ω^4	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ω	0	ω	$\omega 2$	$\cdots$	ω^2	$\omega^2 2$	$\omega^2 3$	$\cdots$	ω^3	ω^4	ω^5	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\omega 2$	0	$\omega 2$	$\omega 4$	$\cdots$	ω^2	$\omega^2 2$	$\omega^2 3$	$\cdots$	ω^3	ω^4	ω^5	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\omega 3$	0	$\omega 3$	$\omega 6$	$\cdots$	ω^2	$\omega^2 2$	$\omega^2 3$	$\cdots$	ω^3	ω^4	ω^5	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ω^2	0	ω^2	$\omega^2 2$	$\cdots$	ω^3	$\omega^3 2$	$\omega^3 3$	$\cdots$	ω^4	ω^5	ω^6	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
ω^3	0	ω^3	$\omega^3 2$	$\cdots$	ω^4	$\omega^4 2$	$\omega^4 3$	$\cdots$	ω^5	ω^6	ω^7	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
ω^4	0	ω^4	$\omega^4 2$	$\cdots$	ω^5	$\omega^5 2$	$\omega^5 3$	$\cdots$	ω^6	ω^7	ω^8	$\cdots$	ω^ω	$\omega^{\omega 2}$	$\omega^{\omega 3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ω^ω	0	ω^ω	$\omega^\omega 2$	$\cdots$	$\omega^{\omega+1}$	$\omega^{\omega+1}2$	$\omega^{\omega+1}3$	$\cdots$	$\omega^{\omega+2}$	$\omega^{\omega+3}$	$\omega^{\omega+4}$	$\cdots$	$\omega^{\omega 2}$	$\omega^{\omega 3}$	$\omega^{\omega 4}$
$\omega^{\omega 2}$	0	$\omega^{\omega 2}$	$\omega^{\omega 2}2$	$\cdots$	$\omega^{2\omega+1}$	$\omega^{\omega 2+1}2$	$\omega^{\omega 2+1}3$	$\cdots$	$\omega^{\omega 2+2}$	$\omega^{\omega 2+3}$	$\omega^{\omega 2+4}$	$\cdots$	$\omega^{\omega 3}$	$\omega^{\omega 4}$	$\omega^{\omega 5}$
$\omega^{\omega 3}$	0	$\omega^{\omega 3}$	$\omega^{\omega 3}2$	$\cdots$	$\omega^{3\omega+1}$	$\omega^{\omega 3+1}2$	$\omega^{\omega 3+1}3$	$\cdots$	$\omega^{\omega 3+2}$	$\omega^{\omega 3+3}$	$\omega^{\omega 3+4}$	$\cdots$	$\omega^{\omega 4}$	$\omega^{\omega 5}$	$\omega^{\omega 6}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
ω^{ω^ω}	0	ω^{ω^ω}	$\omega^{\omega^\omega}2$	$\cdots$	$\omega^{\omega^\omega+1}$	$\omega^{\omega^\omega+1}2$	$\omega^{\omega^\omega+1}3$	$\cdots$	$\omega^{\omega^\omega+2}$	$\omega^{\omega^\omega+3}$	$\omega^{\omega^\omega+4}$	$\cdots$	$\omega^{\omega^\omega+\omega}$	$\omega^{\omega^\omega+\omega 2}$	$\omega^{\omega^\omega+\omega 3}$
$\omega^{\omega^{\omega 2}}$	0	$\omega^{\omega^{\omega 2}}$	$\omega^{\omega^{\omega 2}}2$	$\cdots$	$\omega^{\omega^{\omega 2}+1}$	$\omega^{\omega^{\omega 2}+1}2$	$\omega^{\omega^{\omega 2}+1}3$	$\cdots$	$\omega^{\omega^{\omega 2}+2}$	$\omega^{\omega^{\omega 2}+3}$	$\omega^{\omega^{\omega 2}+4}$	$\cdots$	$\omega^{\omega^{\omega 2}+\omega}$	$\omega^{\omega^{\omega 2}+\omega 2}$	$\omega^{\omega^{\omega 2}+\omega 3}$
$\omega^{\omega^{\omega 3}}$	0	$\omega^{\omega^{\omega 3}}$	$\omega^{\omega^{\omega 3}}2$	$\cdots$	$\omega^{\omega^{\omega 3}+1}$	$\omega^{\omega^{\omega 3}+1}2$	$\omega^{\omega^{\omega 3}+1}3$	$\cdots$	$\omega^{\omega^{\omega 3}+2}$	$\omega^{\omega^{\omega 3}+3}$	$\omega^{\omega^{\omega 3}+4}$	$\cdots$	$\omega^{\omega^{\omega 3}+\omega}$	$\omega^{\omega^{\omega 3}+\omega 2}$	$\omega^{\omega^{\omega 3}+\omega 3}$

Table of Ordinal Exponentiation

^	0	1	2	⋯	ω	$\omega2$	$\omega3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$
0	1	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0
1	1	1	1	⋯	1	1	1	⋯	1	1	1	⋯	1	1	1
2	1	2	4	⋯	ω	$\omega2$	$\omega3$	⋯	ω^2	ω^3	ω^4	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω	1	ω	ω^2	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$	⋯	ω^{ω^2}	ω^{ω^3}	ω^{ω^4}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
$\omega2$	1	$\omega2$	$\omega^2 2$	⋯	$\omega^\omega 2$	$\omega^{\omega2}2$	$\omega^{\omega3}2$	⋯	$\omega^{\omega^2}2$	$\omega^{\omega^3}2$	$\omega^{\omega^4}2$	⋯	$\omega^{\omega^\omega}2$	$\omega^{\omega^{\omega2}}2$	$\omega^{\omega^{\omega3}}2$
$\omega3$	1	$\omega3$	$\omega^2 3$	⋯	$\omega^\omega 3$	$\omega^{\omega2}3$	$\omega^{\omega3}3$	⋯	$\omega^{\omega^2}3$	$\omega^{\omega^3}3$	$\omega^{\omega^4}3$	⋯	$\omega^{\omega^\omega}3$	$\omega^{\omega^{\omega2}}3$	$\omega^{\omega^{\omega3}}3$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^2	1	ω^2	ω^4	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$	⋯	ω^{ω^2}	ω^{ω^3}	ω^{ω^4}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
ω^3	1	ω^3	ω^6	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$	⋯	ω^{ω^2}	ω^{ω^3}	ω^{ω^4}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
ω^4	1	ω^4	ω^8	⋯	ω^ω	$\omega^{\omega2}$	$\omega^{\omega3}$	⋯	ω^{ω^2}	ω^{ω^3}	ω^{ω^4}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^ω	1	ω^ω	$\omega^{\omega2}$	⋯	ω^{ω^2}	$\omega^{\omega^2 2}$	$\omega^{\omega^2 3}$	⋯	ω^{ω^3}	ω^{ω^4}	ω^{ω^5}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
$\omega^{\omega2}$	1	$\omega^{\omega2}$	$\omega^{\omega4}$	⋯	ω^{ω^2}	$\omega^{\omega^2 2}$	$\omega^{\omega^2 3}$	⋯	ω^{ω^3}	ω^{ω^4}	ω^{ω^5}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
$\omega^{\omega3}$	1	$\omega^{\omega3}$	$\omega^{\omega6}$	⋯	ω^{ω^2}	$\omega^{\omega^2 2}$	$\omega^{\omega^2 3}$	⋯	ω^{ω^3}	ω^{ω^4}	ω^{ω^5}	⋯	ω^{ω^ω}	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
ω^{ω^ω}	1	ω^{ω^ω}	$\omega^{\omega^\omega 2}$	⋯	$\omega^{\omega^{\omega+1}}$	$\omega^{\omega^{\omega+1}2}$	$\omega^{\omega^{\omega+1}3}$	⋯	$\omega^{\omega^{\omega+2}}$	$\omega^{\omega^{\omega+3}}$	$\omega^{\omega^{\omega+4}}$	⋯	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega3}}$	$\omega^{\omega^{\omega4}}$
$\omega^{\omega^{\omega2}}$	1	$\omega^{\omega^{\omega2}}$	$\omega^{\omega^{\omega2}2}$	⋯	$\omega^{\omega^{\omega2+1}}$	$\omega^{\omega^{\omega2+1}2}$	$\omega^{\omega^{\omega2+1}3}$	⋯	$\omega^{\omega^{\omega2+2}}$	$\omega^{\omega^{\omega2+3}}$	$\omega^{\omega^{\omega2+4}}$	⋯	$\omega^{\omega^{\omega3}}$	$\omega^{\omega^{\omega4}}$	$\omega^{\omega^{\omega5}}$
$\omega^{\omega^{\omega3}}$	1	$\omega^{\omega^{\omega3}}$	$\omega^{\omega^{\omega3}2}$	⋯	$\omega^{\omega^{\omega3+1}}$	$\omega^{\omega^{\omega3+1}2}$	$\omega^{\omega^{\omega3+1}3}$	⋯	$\omega^{\omega^{\omega3+2}}$	$\omega^{\omega^{\omega3+3}}$	$\omega^{\omega^{\omega3+4}}$	⋯	$\omega^{\omega^{\omega4}}$	$\omega^{\omega^{\omega5}}$	$\omega^{\omega^{\omega6}}$

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