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Abstract. In the early 1900s a rising star in the mathematics world was emerging. I will discuss her life as a female mathematician and the struggles she faced being a rebel in her time. I will also take an in depth look at some of her contributions to the mathematics and science community. Her work in algebra and more specifically, ring theory, are said to be foundations for much of the work done since then. Her developments in abstract algebra helped to unify topology, geometry, logic and linear algebra. Also, Noether's theorem is a widely used theorem in physics along with many other contributions is noteworthy. In short, I will discuss the life of Amalie Emmy Noether and give an in depth analysis of her contribution to ring theory and its application to other mathematics disciplines.

Key words and phrases.

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1. Noether's Life

1.1. **Introduction.** Emmy Amalie Noether was born on March 23, 1882 in Erlangen, Germany. This small university town is found in southern Germany. She was born to Max Noether and Ida Amalia Kaufman. Her mother, Ida, came from a wealthy family in Germany. Emmy's father came from a large family of iron wholesalers. Her family was Jewish, but lived in a time when Germany required their names be Germanic. Originally, Max Noether's grandparents carried the Samuel name. But during the early nineteenth century, a law was set in place to change that. Because of the law, Emmy Noether's great grandfather changed his name to Noether (properly named Nöther) [?]. The iron business would remain in operation until World War II but Max decided to choose a different career path other than the family business. In his early days, Max Noether contracted polio which left him partially disabled. He was a professor of mathematics at the University of Erlangen. Although not as famous as Ms. Noether, her father played a role in the development of the theory of algebraic functions [?]. Young Emmy was not expected to become a great mathematician. She was the older sister of three boys, all of whom died young. Alfred Noether was a doctor of chemistry but died before his career took off. Fritz Noether studied mathematics; however, he was shot due to political reasons by the Soviet Union. The youngest brother Gustav, was physically and mentally disabled his entire life and died in his youth while in an institution [?]. Needless to say, as a woman, Ms. Noether achieved more than her brothers and father. Women at the time were expected to cook, clean, be graceful and lead relatively quiet lives in society. With such a strong example in her father, Emmy did all but stay quiet. She wasn't known for being openly rebellious yet she studied with vigor and proved herself worthy to take her place among the great minds of her time. Ms. Noether lead with brilliance and not brovado.

1.2. **Who was Emmy Noether?** Auguste Dick, an Austrian historian and mathematics teacher who wrote a biography on her life in 1980 says, "Emmy did not appear exceptional as a child. Playing among her peers in the schoolyard on Fahrstrasse she probably was not especially noticeable - a near-sighted, plain-looking little girl, though not without charm. Her teachers and classmates knew Emmy as a clever, friendly, and likeable child. She had a slight lisp and was one of the few who attended classes in the Jewish religion." [**?**] Ms. Noether originally studied language and aspired to be a language teacher. She matriculated through a finishing school for girls and one could speculate that she took this typical path in the beginning as it was expected of a woman. After passing exams, she was ready to begin a career as an English and French instructor. This would be a career she would never begin [OR2].

At the time that Emmy Noether became of age to attend university, women were not allowed. Despite the University of Erlangen's policy prohibiting women from officially matriculating, Emmy decided to audit mathematics courses instead. Since her father taught at the University of Erlangen, Emmy was connected to resources such as private tutors. A professor, Paul Gordan, who just so happened to be a family friend, took on the job as Noether's tutor. He is said to have played a critical role in her mathematical development [O]. Paul Gordan was Constructivist and came under greats such Karl Jacobi, F.B. Reimann, and A. Clebsch. He believed that all mathematical work is constructed by the student and not axiomatic in foundation as Hilbert believed. It is said Gordans work was mainly computational leaving very little explanation for his readers to follow his train of thought. While those he collaborated with were making strides in their respective specialties, Paul Gordan was losing respect amongst his peers. For his efforts to link invariance to chemical valences (how easily atoms or radicals attach to other chemicals), Gordan was heavily criticized by mathematicians and chemists alike. His one and only doctoral student, Emmy Noether, maybe the only salvation to his marred name [F]. Gordan was such an influence to Emmy that she kept a picture of him on her office wall at the University of Erlangen for many years. After Gordan retired she was also tutored by two other mathematicians, Ernst Fischer and Erhard Schmidt. Emmy Noether herself, however credits Fischer for helping change her approach to mathematics: "Above all I am indebted to Mr. E. Fischer from whom I received the decisive impulse to study abstract algebra from an arithmetical viewpoint, and this remained the governing idea for all my later work." Noether was also moiling over Dedikind's theory of modules and ideals at the time. This would help her collaboration with Werner Schmeidler later [ZD]. Because of this work and her dedication, Emmy was creating a name for herself amongst her male counterparts. Even with the praise and recognition, she would not be offered a position with the mathematics department at Erlangen.

Following the death of her mother and her father's retirement, however, at age 21, Ms. Noether was invited to the University of Göttingen by famous mathematician David Hilbert. Since their work was closely related, Hilbert and other colleagues believed Ms. Noether would be a great asset to their team at Gottingen. Emmy would contribute greatly to topics on the theory of relativity and the axiomatic method. At age 25, Ms. Noether completed her doctoral thesis titled "On Complete Systems of Invariants for Ternary Biquadratic Form". Her thesis earned her praise from the mathematics community at the time, but in her own words she called it a jungle of formulas and simply stated,"crap". It was good enough to make a popular mathematical journal, Crelle's Journal. However, her efforts were not good enough for some of her superiors. Being a woman and a Jew put her at a clear and sizable disadvantage. Emmy struggled to find respect amongst her peers and teachers. At the beginning of her time at Göttingen, Emmy was not compensated for teaching. Hilbert fought for her, but did not succeed at first. Although the university was quite ahead of their counterparts in awarding a doctoral degree to a woman, some men could not fathom a woman being equal to them in station. It is said that the professors deciding her fate were not mathematicians; they were philosophers and historians. In their argument, the professors believed that the soldiers would not want to learn from a woman. As told in the book Women In Mathematics, Professor Hilbert had this to say: "Meine Herren, I do not see that the sex of the candidate is an argument against her admission as a Privatdozent. After all, the Senate is not a bathhouse." It turns out that Emmy Noether still lectured but under Hilbert's name. After a few years of doing this and World War I had ended, Emmy was granted "habilitation which allowed her to officially lecture at the university. In 1922, she was given the title of Associate Professor. However, the title carried no benefits or responsibilities and most definitely little to no monetary compensation" [O].

Ms. Noether never married nor had children. She was completely dedicated to her work in mathematics. Of course, this was odd and unusual for her time. Women were supposed to marry, have children and maintain a certain position in society. Acceptable professions for women at the time were early childhood education or language teacher. Other than these, women were expected to be at home with children, in church or cooking up a wonderful meal for husband and household. The early 1900s were an exciting time, however, for women. They would gain the right to vote after much struggle and contention [R]. There was a group called the Blue Stockings that began as a co-ed literary club but became entirely female. During Emmy's time, this club found ways to combat the injustices dealt to especially married women . Women in Germany won the right to vote in 1918. However, a married woman remained voiceless in the eyes of the law. Her husband could divorce her at will taking all possessions with him, but she would be forced to prove he committed horendous acts in order to keep any inheritance she possessed. These injustices could explain why Emmy never married. Ms. Noether was a pacifist and never rallied or openly fought against the injustices against women or Jews unlike the Blue Stockings. One friend Hermann Weyl (also a mathematician) wrote,

> During the wild times after the Revolution of 1918, she did not keep aloof from the political excitement, she sided more or less with the Social Democrats; without being actually in party life she participated intensely in the discussion of the political and social problems of the day. ... In later years Emmy Noether took no part in matters political. She always remained, however, a convinced pacifist, a stand which she held very important and serious. [OR2]

Auguste Dick also writes about a lighter side of Emmy. She would host gatherings in her apartment where students and professors could come to discuss the mathematics of the day or to relax in eating sweets and drinking wine. Though some would argue that without children and a husband, her life was incomplete. In contrast, Emmy had many "children" affectionately named 'Noether's boys'. Later on, I will name a few that had a lasting affect on the mathematics and science communities [ZD].

This did not stop Emmy from being a shining example of being able to change traditional systems with perseverance and dedication. This dedication and perseverance were highly spoken of by colleagues and students alike. It is said that Emmy presented lectures with passion and originality; she was able to explain difficult concepts in simplistic manner so as to be easily understood. Her days at Göttingen were filled with study, teaching and collaboration. In 1920, because of her ground breaking collaboration with Werner Schmeidler, Ms. Noether was finally seen as a great mathematician [ZD]. Also, Ms. Noether mentored and advised some great minds that continued her work. One such student was Grete Hermann; she was one of the only female students Emmy encountered at Göttingen. She continued work on primary decomposition. The first algorithm for computing primary decompositions for polynomial rings was published by Grete Hermann. She actually studied at Göttingen under Emmy and completed her doctoral thesis in 1926, The Question of Finitely Many Steps in Polynomial Ideal Theory. Unlike her predecessor, Grete did not continue in mathematics after she was pushed out of Germany by the Nazi regime. Philosophy and quantum mechanics are the main areas for which she is most noted. To name a few less known advisees: Hans Falckenberg in 1911, Fritz Seidelmann in 1916, Heinrich Grell 1927, Werner Weber in 1930 (he joined the Nazi party after being awarded his doctorate), Jakob Levitski in 1929 (most famous for the Leviski's theorem), Max Deuring in 1930 (one of her most successful students who was able to, like Emmy, generalize and simplify existing research.), Ruth Stauffer in 1935 (at Bryn Mawr, Ms. Noether would not see her conferred due to her sudden death. Ruth was a teacher but later in life became an analyst.) [OR2]

While Emmy Noether continued to gain notoriety and acclaim for her research and collaborative writings, the Nazi uprising would soon threaten not only her position at the university, but her life. By the time the Nazis had full control in 1933, it was clear that Emmy would no longer be able to remain at the university. Not only was Emmy a Jew, but also a woman and liberal, which added to her speedy termination as a professor. Though surely this was a devastating blow to her hard work,with the help of Albert Einstein Ms. Noether would be invited to Bryn Mawr and then to the Institute of Advanced Study both of which are in Princeton, New Jersey. According to Bryn Mawr College (who at the time was predominantly a female college), Emmy taught there under an Emergency Committee in Aid of Displaced Scholars. This group was formed to support Jewish refugees during WWII and place them in American institutions. She once again found great admiration and respect in these places, her reputation preceding her. Emmy inspired her students and coleagues alike by showing a relentless dedication to mathematics. Ms. Noether took one last trip back to Germany in 1934 to visit her brother Fritz and another mathematician Emil Artin (they shared countless discussions on noncommutative rings). Artin's wife is quoted as saying that Emmy spoke freely and loudly intertwining politics and mathematics. In a time when the Nazi's were gaining strength and terror, Artin's wife grew nervous of her loud and excited conversation! Emmy Noether soon returned to Bryn Mawr as her professorship was extended for another year. Unfortunately, her career would be cut short at these institutions. Doctors discovered uterine tumors after she fell ill and an operation to remove them was immediately carried out. Though her condition improved for a few days, she turned feverishly ill and died suddenly. [OR2]

On May 4, 1935, Albert Einstein was quoted as saying, "Fraulein Noether was the most significant, creative, mathematical genius thus far produced since the higher education of women began".[**?**] He also stated that her discoveries in algebra were of the utmost importance to the new generation of mathematicians. Hermann Weyl again addressed her lasting impact on others at her funeral, "You did not believe in evil, indeed it never occurred to you that it could play a role in the affairs of man. This was never brought home to me more clearly than in the last summer we spent together in Gttingen, the stormy summer of 1933. In the midst of the terrible struggle, destruction and upheaval that was going on around us in all factions, in a sea of hate and violence, of fear and desperation and dejection - you went your own way, pondering the challenges of mathematics with the same industriousness as before. When you were not allowed to use the institute's lecture halls you gathered your students in your own home. Even those in their brown shirts were welcome; never for a second did you doubt their integrity. Without regard for your own fate, openhearted and without fear, always conciliatory, you went your own way. Many of us believed that an enmity had been unleashed in which there could be no pardon; but you remained untouched by it all."

This great woman is not well known to the world, but some physicists and mathematicians have credited her with possibly being the greatest mind in the 20th century. The New York Times wrote a piece about her in 2012 and included a physicists point of view and research on the matter. Ms. Noether was honored with a few awards in her time: Speaker at International Congress in 1932, lunar features (a crater named after her called Crater Noether). Dave Goldberg of Drexel University took a poll to discover that even amonst his colleagues and students, very few knew her name let alone her contributions to the science and mathematics communities. If there were more men like David Hilbert and Felix Klein to support Ms. Noether, she would have, without question, contributed much more to a degree that is unmeasurable. [OR2]

1.3. **Timeline of Emmy Noether's life and work.**

1882:

Noether was born on March23 .

1900:

She learned English and French and attended Mathematics classes from the University of Erlangen.

1903:

She joined the University of Göttingen.

1907:

She received her Doctorate degree in Mathematics from University of Erlangen.

1915:

She joined the Mathematics department at the University of Göttingen.

1921:

She published Theory of Ideals in Ring Domains.

1932:

She was awarded the Ackermann-Teubner Memorial Prize in Mathematics.

1933:

She became a guest professor at Bryn Mawr College in Pennsylvania, U.S.

1935:

Noether died on April 14 , from the complications that followed a uterine surgery.

1.4. **Noether's Works.** To give you a progression of Emmy Noether's notable work, we have:

(1) Doctoral Thesis - On Complete Systems of Invariants for Ternary Biquadratic Forms

(2) Noether Factor Sets

(3) Noether's Problem in Galois theory

(4) Noether Normalization

(5) Non-commutative Methods in Algebraic Number theory

(6) Conservation Laws and their application in Global Differential Geometry (Noether's Theorem)

(7) Finite Simple Groups

(8) The Study of linear associative algebras (in the United States for this)

It is important to note that Emmy continued her father's work in generalizing some of his theorems. Ms. Noether also collaborated with several mathematicians to publish several papers. Her collaborations included work with Helmut Hasse and Richard Brauer on noncommutative algebras, Werner Schmeidler on moduli of noncommutative fields particularly in differential and difference terms. She is also known for taking previously proved theorems and improving them or generalizing them as the Lasker-Noether theorem that will be discussed later on. Emmanuel Lasker was a mathematician and chess master who was world champion for 27 years. Lasker actually has an interesting connection to Ms. Noether. He was advised by her father Max Noether, presenting his thesis to the University of Erlangen in 1900. He was living in the United States when he introduced primary ideals which lead to his most notable work in primary decomposition. Undoubtedly, Emmy must have interacted with him while he was under the supervision of her father or at least been privy to his work.

Noether's work was divided into 3 periods. The first period was between 1907-1919, in which she devoted her time in the field of algebraic invariant theory, Galois Theory and physics. Noether proved two theorems that were important for elementary particle physics and general relativity. One of her theorems known as Noethers Theorem is one of the most significant contributions in the development of modern physics. This is an extremely powerful theorem used in physics which states that the symmetry of an object leads to a physically conserved quantity. I would be remiss if I did not briefly share such a contribution of this stature. Here are the three main points:

(1) Symmetry under translation corresponds to conservation of linear momentum

(2) Symmetry under rotation corresponds to conservation of angular momentum

(3) And symmetry in time corresponds to conservation of energy.

The Encyclopedia Brittanica puts it plainly that the conservation laws of (classical) physics are several principles that state that certain physical properties do not change in the course of time within an isolated physical system. Meaning that when a body or system of bodies retains its total momentum (the product of mass and vector velocity) linear momentum is conserved unless something outside of the system acts on it. Similiarly, a body or system of bodies retains its rate while rotating unless torque is applied to it. Last but not least, is the energy conservation law where energy is not produced nor lost in a system but can be changed from one state to another. In an isolated system the sum of all energy forms remains the same. Noether's Theorem is a generalization of these laws that she cleverly put together using a higher order Lagrangian (a Lagrangian, named after Joseph Lagrange, is a special funtion used in optimization theory).

In the second period from 1920-1926, she concentrated on the theory of mathematical rings. She developed the abstract and conceptual approach to algebra, which resulted in several foundational principles unifying topology, logic, geometry, algebra and linear algebra. Her works were a breakthrough in abstract algebra and had mathematicians from around the world asking for her input in their work. During this time she also axiomatized the general theory of ideals for all cases. Ms. Noether's study based on chain conditions on the ideals of commutative rings were honored by many mathematicians all over the world. Her paper 'Idealtheorie in Ringbereichen or 'Theory of Ideals in Ring Domains, published 1921, became the foundation for commutative ring theory. The Noetherian rings and Noetherian ideals formed part of her mathematical contributions. Her insights and ideas in topology had a great impact in the field of mathematics as well.

The third period began from 1927-1935, where non-commutative algebras (the study of rings that do not commute over multiplication), representation theory (the study of abstract algebraic structures by representing them as linear transformations of vector spaces and studies modules over these abstract algebraic structures), hyper-complex numbers(numbers outside of the real and complex numbers, like Quaternions with applications in three dimensional rotations as in computer graphics) and linear transformations became the primary focus of her study. Noether was awarded the Ackermann-Teubner Memorial Prize in Mathematics in 1932 for this work.

As Ms. Noether's works are extensive, I will only discuss her contribution to commutative algebra for brevity.

2. Noether's Contributions to Commutative Algebra and Applications in Algebraic Geometry

2.1. **What is a commutative ring and what is an ideal?** Emmy's work on Factor Sets and Normalization were results from her famous Ascending Chain Condition (A.C.C.). A.C.C. gives leverage to prove many things about finitely generated rings. When Emmy introduced the ascending chain condition on a ring, it was sufficient to show finiteness. It is important to note that the most interesting rings in commutative algebra are in fact Noetherian rings. Then let us define some terms needed to explain the A.C.C. We will need a commutative ring which is an algebraic structure consisting of a set together with two binary operations usually called addition and multiplication, where the set is an abelian group under addition, addition and multiplication are associative, multiplication distributes over addition, and where multiplication is commutative. In addition, we will be discussing chains of ideals, so it is prudent to define ideals as they relate to rings. An ideal is a special subring of a given ring that is nonempty, closed under multiplication by all of the elements of the ring. A simple example of a commutative ring with ideals satisfying A.C.C. is following: Let the ring R be the set of integers $\mathbb{Z}$, where $2\mathbb{Z} = I$ an ideal of $\mathbb{Z}$. We know that $2 \in I$ and $-3 \in R$ implies $-3(2) = -6 \in I$.

2.2. **Noether's Ascending Chain Condition.** Here is the ascending chain condition in a nutshell:

Definition. A commutative ring R is said to be Noetherian if there is no infinite increasing chain of ideals in R, i.e. whenever $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ is an increasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Theorem 1. From this comes 3 equivalent statements:

(a) R is a Noetherian Ring

(b) Every nonempty set of ideals, I of R contains a maximal element. (A maximal element of R is an ideal $I' \neq R$ and the only ideals containing I' are I' and R.)

(c) Every ideal I is finitely generated.

The forward proof is as follows:

$(a \Rightarrow b)$

Assume R is Noetherian and let S be a nonempty set of ideals of R. Then choose $I_1 \in S$. If I_1 is a maximal element of S then (b) holds. So assume that I_1 is not maximal. Hence there must be an $I_2 \in S$ such that $I_1 \subset I_2$. If I_2 is maximal then (b) holds. Assume there exists $I_3 \in S$ such that $I_1 \subset I_2 \subset I_3$. If (b) fails, then there is an infinite increasing chain of elments of S, contrary to (a).

$(b \Rightarrow c)$

Assume (b) holds and let $I \subset R$ be an ideal. Then let S be the set of all finitely generated ideals of I. Since $0 \in S$, S is not empty. By (b), S also has a maximal element I_1. We can clearly see that $I_1 \subset I$. If $I_1 \neq I$, let $x \in I - I_1$. From our assumption that $I_1 \in S$ we see that I_1 is finitely generated. Hence the ideal ${I_1}'$ generated by I_1 and x is also finitely generated. Then by the definition of S, ${I_1}' \in S$.This is a contradiction to the maximality of I_1, since $I_1 \subsetneq {I_1}'$. Therefore, $I = I_1$ is finitely generated.

$(c \Rightarrow a)$

Assume (c) holds and let $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ be a chain of ideals in R. Also let,

$$J = \bigcup_{n=1}^{\infty} I_n$$

then J is an ideal of R. By (c), J is generated by $a_1, a_2, a_3, ..., a_n$. Since $a_i \in J$, $a_i \in I_{j_i}$ for some j_i. Let $m = max[j_1, ..., j_n]$. So $a_i \in I_m$ for all i. Therefore, $J = (a_1, a_2, a_3, ..., a_n) \subset I_m$. This implies $I_m = J = I_k$ for all $k \geq m$, which proves (a).□

The last proof, $(c \Rightarrow a)$, shows that if the ideals are finitely generated, then the chain of ideals in R will stabilize (or terminate due to the maximal element in our set of ideals).

A subsequent propostion states that if I is an ideal of a Noetherian ring R, then the quotient R/I is also Noetherian. In the proof we would obviously use A.C.C.There are many applications to these results and here are a few.

2.3. **Basic Algebraic Geometry: Affine Varieties.** Algebraic Geometry had its notable beginnings with Pierre de Fermat and Rene Descartes between 1630 and 1795. These mathematicians and others, used geometry to explain algebraic problems. According to *Abstract Algebra* by David Summit and Richard Foote, the general idea behind algebraic geometry is to equate geometric questions with algebraic questions involving ideals in rings. By the time David Hilbert and Emmy Noether came along, more theoretical work was advancing. Of course, Ms. Noether reduced questions concerning these rings by only considering finitely generated ideals of given rings. This is the motivation that brought forth the following theorem.

Theorem 2. (Hilbert's Basis Theorem) If a ring R is Noetherian, then so is the polynomial ring $R[x]$.

We begin the proof by letting I be an ideal in $R[x]$ and let L be the set of leading coefficients of the elements in I. We first show that L is an ideal of R. Since I contains the zero polynomial, $0 \in L$. Let $f(x) =$

$ax^d + a_{d-1}x^{d-1} + ... + a_1x + a_0$ and $g(x) = bx^e + b_{e-1}x^{e-1} + ... + b_1x + b_0$ be polynomials in I of degrees d and e respectively, and leading coefficients a, $b \in R$ respectively. Then for any $r \in R$ either $ra - b$ is zero (zero in R) or it is the leading coefficient in the polynomial $rx^e f - x^d g$. This should be apparent once we break down f and g in L. The following equation shows succinctly, how we arrived at the answer:

$rx^e f - x^d g = rx^e[ax^d + ...] - x^d[bx^e + ...]$

Since $rx^e f - x^d g$ is a polynomial in I, we know that $ra - b \in L$, which shows L is an ideal of R. We know that R is Noetherian so $L \in R$ is finitely generated by some $c_1, c_2, ..., c_n \in R$ which are all the leading coefficients of the polynomials. For each $i = 1, ..., n$, let f_i be an element of I whose leading coeffecient is a_i. Let e_i denote the degree of f_i, and let N be the maximum of $e_1, e_2, ..., e_n$.

For each $d \in \{0, 1, 2, ..., N-1\}$, let L_d be a set of all leading coefficients of polynomials in I of degree d together with 0. A similar argument as that for L shows each L_d is also an ideal of R, again finitely generated since R is Noetherian. For each nonzero ideal L_d let $b_{d,1}, ..., b_{d,n_d} \in R$ be a set of generators for L_d and let $f_{d,i}$ be a polynomial in I of degree d with leading coefficient $b_{d,i}$. We show that the polynomials $f_1, ..., f_n$ together with all the polynomials $f_{d,i}$ for all the nonzero ideals L_d are a set of generators for I, i.e. that

$$I = (\{f_1, ..., f_n\} \bigcup \{f_{d,i} | 0 \leq d < N, 1 \leq i \leq n_d\}).$$

By construction, the ideal I' on the right above is contained in I since all the generators were chosen in I. If $I' \neq I$, there exists a nonzero polynomial $f \in I$ of minimum degree with $f \notin I'$. Let $d = degf$ and let a be the leading coefficient of f.

Suppose first that $d \geq N$. Since $a \in L$ we may write a as an R - linear combination of the generators of $L : a = r_1a_1 + ... + r_na_n$. Then $g = r_1x^{d-e_1}f_1 + ... + r_nx^{d-e_n}f_n$ is an element of I' with the same degree d

and the same leading coefficient a as f. Then $f - g \in I$ is a polynomial in I of smaller degree than f. By the minimality of f, we must have $f - g = 0$, so $f = g \in I'$, a contradiction.

Suppose next that $d < N$. In this case $a \in L_d$ for some $d < N$, and so we may write $a = r_1 b_{d,1} + ... + r_{n_d} b n_d$ for some $r_i \in R$. Then $g = r_1 f_{d,1} + ... + r_{n_d} f_{n_d}$ is a polynomial in I' with the same degree d and the same leading coefficient a as f, and we have a contradiction as before.

It follows that $I = I'$ is finitely generated, and since I was arbitrary, this is the proof that $R[x]$ is Noetherian.$\square$ [DF]

Examining Hilbert's Basis Theorem, the proof uses the equivalences from (c) to (a) to show $R[x]$ is Neotherian. We see a straight forward progression from the coefficients of our f_i polynomials equal to some ideal of R which is finitely generated, to the polynomials f_i in some ideal of $R[x]$ which we can clearly see now is finitely generated. Hence, finitely generated ideals leads to a Noetherian ring $R[x]$.

An affine space $\mathbb{A}^n$ is the set of n-tuples of elements of a field k. A little later, we will discuss algebraic sets in affine spaces which denote a locus of zeros of a collection of polynomials in the space. Affine sets were initially a part of algebraic geometry, defining affine spaces by polynomial equations.

Since a field is Noetherian, this leads us to a corollary to Hilbert's Basis theorem. The polynomial ring $k[x_1, x_2, ..., x_n]$ is a Noetherian ring, where k is a field. To begin, we must define a field. A field k, is a special nonzero ring that is abelian under multiplication and every nonzero element has an inverse with respect to multiplication. Here is an example for this new corollary. Let $f_1, f_2, ..., f_m, ... \in \mathbb{C}[z_1, ..., z_n]$ be polynomials over the complex plane in n variables $z_1, ..., z_n$. Consider the common zero set of these polynomials:

$$Z(f_1, ..., f_m, ...) = \{z = (z_1, ..., z_n) \in \mathbb{C}^n | f_i(z) = 0, i = 1, 2, 3, ...\}.$$

Let $I = (f_1, f_2, ..., f_m, ...)$ be the ideal generated by $f_1, f_2, ...f_m, ...$ Since $\mathbb{C}[z_1, ..., z_n]$ is Noetherian, I must be generated by finitely many polynomials $f_{i1}, ..., f_{im}$. Hence, $I = (f_1, f_2, ..., f_m)$. We may assume that any $m-1$ polynomials among $f_1, ..., f_m$ do not generate I.

Then,

$$(f_1) \subseteq (f_1, f_2) \subseteq \cdots \subseteq (f_1, ..., f_m) = (f_1, ..., f_m, f_{m+1}) = \cdots$$

If $l \in \mathbb{N}$, then there exists $g_1, ..., g_m \in \mathbb{C}[z_1, ..., z_n]$ such that $f_{m+l} = f_1g_1 + f_2g_2 + ... + f_mg_m$. If $f_1(z) = ... = f_m(z) = 0$, then $f_{m+l}(z) = 0$ for all $l \in \mathbb{N}$. So the zero set $Z(f_1, ..., f_m, f_{m+1}, ...) = Z(f_1, ..., f_m)$. The ideal $(f_1, ..., f_m)$ determines the zero set $Z(f_1, ..., f_m) = V$, called the affine algebraic set defined by $f_1, ..., f_m$ in $\mathbb{C}^n$. V is irreducible if it has only one component or it cannot be written as $V = V_1 \bigcup V_2$, where V_1 and V_2 are proper algebraic sets in V.

Our next theorem is comprised of two parts:

Theorem 3. (1) The affine set V is irreducible if and only if

$$I(V) = \{f \in k[x_1, ..., x_n], f(p) = 0, \forall p \in V\}$$

is a prime ideal. ($I \subset R$ is a prime ideal if $a, b \in R$, $ab \in I$, then $a \in I$ or $b \in I$)

(2) If $V \neq \phi$, then V can be uniquely written in the form $V = V_1 \bigcup V_2 \bigcup ... \bigcup V_g$ where each V_i is irreducible, and $V_i \not\subseteq V_j$ for all $j \neq i$.

From this theorem we know that every algebraic set may be uniquely decomposed into a finite union of irreducible components.

Example 1. Let $f(z) \in \mathbb{C}[z]$ be a polynomial with a single variable $z = x + iy$, where $x, y \in \mathbb{R}$, $i^2 = -1$. Then $Z(f)$ is an algebraic set with finitely many points in $\mathbb{C}$, because by the Fundemental Theorem of Algebra we can write,

$f(z) = c(z-z_1)^{m_1}(z-z_2)^{m_2}...(z-z_g)^{m_g}$, where $m_1, ..., m_g \in N$ and $z_1, ..., z_g \in \mathbb{C}$.

Example 2. Let $h(z) = \sin z, z \in \mathbb{C}$. Then $\sin z$ is a holomorphic or differentiable function at all points in $\mathbb{C}$ but not a polynomial. The zero set $Z(\sin z) = \{z \in \mathbb{C}, \sin z = 0\} = \{(k\pi, 0), \forall k \in \mathbb{Z}\}$ is not a finite set.

Here is one more example of the uses of the Ascending Chain Condition. This example incorporates Noetherian rings with topology in a very succinct and familiar way. Since topology is the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures, we define a new space called a Noetherian Topological Space. These spaces incorporate the same foundational axioms used to define a topological space (a set of points, along with a set of neighborhoods for each point, that satisfy a set of axioms relating points and neighborhoods).

A topological space X is Noetherian if the closed subsets of X satisfy the descending chain condition. This is true for any sequence of the form $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$ where closed subsets $Y_i \in X$ and for any integer n we have $Y_n = Y_{n+1} = Y_{n+2} = \cdots$.

Definition. A subset X of $\mathbb{A}^n$ over field k is an algebraic set if there is a subset $B \subset k[x_1, x_2, ..., x_n]$ such that

$$X = Z(B) = \{P \in \mathbb{A}^n, f(P) = 0, \forall f \in B\}$$

.

We can show that

(a) the union of two algebraic sets in $\mathbb{A}^n$ is an algebraic set: if $Y_1 = Z(B_1)$ and $Y_2 = Z(B_2)$, then $Y_1 \cup Y_2 = Z(B_1B_2)$, where $B_1B_2 = \{fg, f \in B_1, g \in B_2\}$.

(b) the intersection of any family of algebraic sets is an algebraic set: if $Y_\alpha = Z(B_\alpha)$ is any family of algebraic sets, then $\cap Y_\alpha = Z(\cup_\alpha B_\alpha)$.

(c) the empty set $\emptyset = Z(1)$ is an algebraic set and $\mathbb{A}^n = Z(\emptyset)$ is an algebraic set.

In $\mathbb{A}^n$, we define a closed subset to be an algebraic set and an open subset to be the complement of an algebraic set. Then we have a topology, called Zariski topology in $\mathbb{A}^n$: the intersection of two open subsets is open and the union of any family of open subsets is open.

Example. $\mathbb{A}^n$ with Zariski topology is a Notherian topological space: for any sequence

$$Y_1 \supseteq Y_2 \supseteq ...$$

of Zariski closed subsets, we have a chain of their ideals

$$I(Y_1) \subseteq I(Y_2) \subseteq ...$$

Since $k[x_1, ..., x_n]$ is a Noetherian space, there is an integer r such that $I(Y_r) = I(Y_{r+1}) = ...$ Now for each i, $Y_i = Z(I(Y_i))$, so the descending chain of the closed sets Y_i is also stationary (or will stabilize).

2.4. **Noether's Normalization Theorem.** Noether's Normalization Theorem applies techniques from algebraic theory to affine geometry. So in order to make better sense of this powerful theorem, we will introduce some preliminary definitions and examples.

(1). Definition. a) Let A be a subring of commutative ring B ($A \subset B$). Let $\alpha \in B$. The element α is said to be integral over A if α is a root of a polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ with coefficients $a_i \in A$ and degree $n \geq 1$.

b) B is integral over A if every element of B is integral over A.

(2) Definition. Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism $f : R \to A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of $A = \{y \in A | \ \forall \ a \in A, \ ya = ay\}$.

(3) Defintion. If k is a field, the elements $y_1, y_2, ..., y_q$ in some k-algebra are called algebraically independent over k if there is no nonzero polynomial

$P(x_1, x_2, ..., x_q)$ in q variables over k such that $P(y_1, y_2, ..., y_q) = 0$. A quick example of this concept is seen when looking at the field $\mathbb{Q}$ of rational numbers. We know that $\sqrt{\pi}$ is a transcendental number and that the set $\{\sqrt{\pi}\}$ is not a root of any nontrivial polynomial whose coefficients are rational numbers.

Example 1. Let R be a commutative ring with identity 1. Then the following must be true:

a. R is a $\mathbb{Z}$ - algebra

b. The polynomial ring $R[x_1, ..., x_n]$ with n variables is a R - algebra

Example 2. Consider k - algebra $k[x]$, where k is a field. Then x, x^2 are algebraically dependent since they satisfy y_1^2 - $y_2 = 0$. But y, x are algebraically independent in $k[x, y]$.

(3) Definition. Any R - algebra is finitely generated if it is finitely generated as a ring over $f(R)$.

Example 3. Consider $R[x, y]$. Given x and y are algebraically independent over R. But x, y, and $x + y$, are algebraically dependent over $R[x, y]$.

Example 4. $R[x+y, x-y, x^2-y^2] = R[x+y, x-y]$ is a finitely generated algebra over R.

Now that we have defined some key terms we are ready to introduce the Noether Normalization Theorem.

Noether Normalization Theorem.

Theorem 4. Let k be a field and suppose that $A = k[r_1, r_2, ..., r_m]$ is a finitely generated k-algebra. Then for some q, $0 \leq q \leq m$, there are algebraically independent elements $y_1, y_2, ..., y_q \in A$ such that A is integral over $k[y_1, y_2, ..., y_q]$.

Proof.

If $r_1, ..., r_n$ are algebraically independent over k, then let $y_i = r_i$, $i = 1, 2, ..., n$. In this case, we are done. Otherwise, assume $r_1, ..., r_n$ are algebraically dependent. Then there exists $f(x_1, ..., x_n) \in k[x_1, ..., x_n]$ such that

$f(r_1, ..., r_n) = 0$. Write $f(r_1, ..., r_n) = 0$ as $\sum a_{(j)}\ r_1^{j1} \cdots r_n^{jn} = 0$, where $\forall a_{(j)} \in k, a_{(j)} \neq 0$ and $(j) = (j_1, ..., j_n)$. Let $m_2, ..., m_n \in N$ and put $y_2 = r_2 - r_1^{m_2}, ..., y_n = r_n - r_1^{m_n}$. Substitute $r_i = y_i + r_1^{m_i}$ in the above equation.

Hence we have,

$$\sum a_{(j)} r_1^{j_1}(y_2 + r_1^{m_2})^{j_2} \cdots (y_n + r_1^{m_n})^{j_n} = \sum c_j r_1^{j_1 + m_2 j_2 + \cdots + m_n j_n} + h(r_1, y_2, ..., y_n),$$

where h is a polynomial in which no pure power of x_1 appears. We now select d to be a large integer [greater than any component of a vector (j) such that $c_{(j)} \neq 0$] and take $(m) = (1, d, d^2, ..., d^n)$. Then all $(j)(m) = (j_1, ..., j_n)(1, d, ..., d^n) = j_1 + m_2 j_2 + \cdots + m_n j_n$ are distinct for those j such that $c_j \neq 0$. In this way, we obtain an equation for r, over $k[y_2, ..., y_n]$. Since each r_i $(i > 1)$ is integral over $k[y_2, ..., y_n]$. We can see how that $k[r_1, ..., r_n]$ is integral over $k[r_2, ..., r_n]$.□

Example 5. k is a field and $k \subset k[x] \subset k[x, y]$. x is not integral over k. y is not integral over $k[x]$. But x is integral over $k[x]$ since it satisfies the equation $X - x = 0$.

(5) Definition. A field k is algebraically closed if every polynomial with coefficients is k has a root in k.

Example 6. $\mathbb{Q}$, $\mathbb{R}$ are not algebraically closed. $\mathbb{C}$ is an algebraically closed field by the Fundemental Theorem of Algebra.

(6) Definition. Let M be an ideal in a given ring R. M is a maximal ideal if $M \neq R$ and the only ideals containing M are M and R.

Here is another useful biproduct of Noether's Normalization Theorem: If R is a commutative ring with identity I, then M is a maximal ideal of R if and only if R/M is a field.

2.5. **The Lasker-Noether Theorem on Primary Decompostion.** The chess master, Emanuel Lasker proved that in a polynomial ring, every ideal is a finite intersection of primary ideals. This began with the narrow view of polynomial rings and convergent power series rings. Emmy Noether proved that the theorem is true for any Noetherian ring which widened the scope of the original theorem.

Definition. An ideal I in a ring R is irreducible if I is not a finite intersection of ideals strictly containing it, i.e. I cannot be written as $I = \bigcap_{j=1}^{n} I_j$; $I \subset I_j$, where $j = 1, ..., n$ and all I_j are ideals of R.

Here is a lemma that is closely related to the Lasker-Noether Theorem: Lemma 1. In a Noetherian ring R, every ideal is a finite intersection of irreducible ideals.

Proof. Suppose that there is an ideal $I \subset R$ such that I is not a finite intersection of irreducible ideals. Let S be the set of all ideals of R which are not finite intersections of irreducible ideals. By assumption, S is nonempty. Since R is a Noetherian ring, by the equivalent statements from the beginning that asert the A.C.C., S has a maximal element M under inclusion:

(a) R is a Noetherian Ring

(b) Every nonempty set I of R contains a maximal element

(c) Every ideal I is finitely generated.

Since M cannot be irreducible, there are two ideals M_1 and M_2 in R such that $M = M_1 \cap M_2$, $M \subset M_1$, $M \subset M_2$. Since M is a maximal element in S, by definition of S, M_1 and M_2 are finite intersections of irreducible ideals. This forces M to be a finite intersection of irreducible ideals. This a contradiction.$\square$

Definition. Let I, J be ideals of the commutative ring R. We define the ideal quotient $(I : J)$ by $(I : J) = \{a \in R | aJ \subseteq I\}$. This is another ideal of R and $I \subset (I : J)$.

Definition. A proper ideal I (meaning $I \neq$ ring R) of a commutative ring R is called primary if $ab \in I$, $a \notin I$, then $b^n \in I$ for some $n \in \mathbb{N}$.

Lemma 2. In a Noetherian ring R, every irreducible ideal is primary.

Proof. Let Q be an ideal of R. Suppose that Q is not primary, that is, $\exists b, c \in R$, $b \notin Q$, $c \notin Q$ but $bc \in Q$ and no power of b lies in Q. Let (b^j) be the ideal generated by b^j and

(1) $I_j = (Q : (b^j)) = \{x \in R | x(b^j) \subset Q\}$. Then we have an increasing chain $I_1 \subset I_2 \subset I_3 \subset \cdots$

By A.C.C., $\exists n \in \mathbb{N}$, such that $I_n = I_{n+1}$, i.e., $[Q : (b^n)] = [Q : (b^{n+1})]$.

We will show that $Q = (Q + Rb^n) \cap (Q + (c))$.

It is clear that $(Q+Rb^n) \cap (Q+(c)) \supset Q$. Let $x \in (Q+Rb^n) \cap (Q+(c))$. Then $\exists u, v \in Q$, $z \in R$, $m \in \mathbb{Z}$, such that $x = u + yb^n = v + zc + mc$. Since $bc \in Q$, we have $bx = bv + zbc + mbc \in Q$ and $yb^{n+1} = bx - bu \in Q$. From $[Q : (b^n)] = [Q : (b^{n+1})]$, we have $\{r \in R | r(b^n) \subset Q\} = \{s \in R | s(b^{n+1}) \subset Q\}$. So $yb^n \in Q$ since $y \in [Q : (b^{n+1})]$. By $x = u + yb^n = v + zc + mc$ we get $x = u + yb^n \in Q$. This shows that $(Q + Rb^n) \cap (Q + (c)) \subset Q$. Bringing together $(Q + Rb^n) \cap (Q + (c)) \supset Q$ and $(Q + Rb^n) \cap (Q + (c)) \subset Q$ we have $Q = (Q + Rb^n) \cap (Q + (c))$. Now $(Q \subsetneqq Q + (c))$ since $c \notin Q$ and $(Q \subsetneqq Q + Rb^n)$ since $b^{n+1} \in Rb^n$, $b^{n+1} \notin Q$. So Q is not irreducible by $Q = (Q + Rb^n) \cap (Q + (c))$. This is a contradiction and Q is primary.□

The Lasker-Noether Decomposition Theorem.

Theorem 5. In a Noetherian ring R, every ideal admits an irredundant representation as finite intersection of primary ideals.

Proof. A representation $Q = \bigcap_i Q_i$ of an ideal $Q \subset R$ as an intersection of primary ideals Q_i is irredundant (or reduced) if it satisfies the following conditions:

(a) No Q_i contains the intersection of the other ones: $Q_i \not\supset \cap_{j \neq i} Q_j$

(b) $\sqrt{Q_i} \neq \sqrt{Q_j}$ where the associated prime $\sqrt{Q_i}$ is the radical $\sqrt{Q_i} = \{x \in R | x^n \in Q_i$ for some $n \in \mathbb{N}\}$ (Hence, if I is a primary ideal, then the radical $\sqrt{I} = \{x \in R | x^n \in I$, for some $n \in \mathbb{N}\}$ is a prime ideal.)

By Lemma 1 and Lemma 2, any ideal I in R has a finite primary decomposition $I = \cap_i Q_i$. We can find an irredundant decomposition as follows:

First, we group together all the Q_i which have the same radical $\sqrt{Q_i} = P_j$ and take their intersection Q'_j, which is primary for P_j. Then $Q = \cap_j Q'_j$ and if some Q'_j contains the intersection of the others, we omit it and proceed in the same way until condition (a) is satisfied.□

2.6. **Applications in Algebraic Geometry.** So based upon the Lasker-Noether Theorem, the following lemma was formed: A prime ideal P is irreducible.

Proof. If P is not irreducible, then there are two ideals I_1 and I_2 such that $P \subset I_1$, $P \subset I_2$ and $P = I_1 \cap I_2$. Hence $I_1 \neq I_2$. Choose $x_1 \in I_1$, $x_1 \notin I_2$ and $x_2 \in I_2$, $x_2 \notin I_1$, then $x_1x_2 \in I_1$ and $x_1x_2 \in I_2$ since I_1 and I_2 are ideals. So it must be that $x_1x_2 \in P$, but $x_1 \notin P$ and $x_2 \notin P$. This contradicts the fact that P is a prime ideal. So P must be irreducible.□

Example 1. The maximal ideal M is a commutative ring R with identity 1 is a prime ideal so irreducible.

Example 2. A prime ideal P is primary.

Example 3. A primary ideal may not be irreducible. Let $R = k[x, y]$, k is a field. Let $M = (x, y)$, a maximal ideal. Then $M^2 = (x^2, xy, y^2)$ is a prime ideal by the following propostions:

(1) Prime ideals are primary.

(2) M is a primary ideal if and only if every zero divisor R/M is nilpotent.

(3) If M is primary, then $radM$ is a prime ideal, and is the unique smallest prime ideal containing M.

(4) If M is an ideal whose radical is a maximal ideal, then M is primary.

But $M^2 = (M^2 + Rx) \cap (M^2 + Ry)$. So M^2 is not irreducible.

3. Conclusion

The life and times of this great mathematician, Emmy Amalie Noether, were full of excitement and danger. But somehow, according to her friends and colleagues, Ms. Noether remained unchanged and diligent in her search for problems to pose and solve. She was not selfish in this endeavor, needless to say; she encouraged and guided those who took the time to know her and ask for help. Loud and boisterous, yet humble and giving are words that most use to describe Emmy. Though she did so much in her short life span, we have only discussed Emmy Noether's work in Commutative Algebra along with some extensions that her work afforded other mathematicians. Her collaborations were note-worthy as well, speaking to the fact that not all men were concerned with Emmy's womanhood. Her colleagues stayed faithful to her cause of simply earning the wages due to any deserving faculty member. With great care, we have introduced Ascending Chain Condition which lends to applications in algebraic geoemtry like Hilbert's Basis Theorem with Affine Sets, Noether's Normalization Theorem, and the Lasker-Noether Decompostion Theorem. Breifly, we made the connection between Noether's Theorem and physics. Its importance was and is stressed by many scientists and even Albert Einstein paid attention to this theorem and how it supported the theory of relativity. She is still gaining respect and acclaim today as others, including myself, are coming to know the genius and range of such a woman. Ms. Noether will never know the magnitude of her contributions, unfortunately. However, for the countless women she has and will inspire, we now can take her example of unwavering diligence and strength to help another overcome the prejudices women still face today.

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