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An Introduction to the Lebesgue Integral

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AN INTRODUCTION TO THE LEBESGUE INTEGRAL

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Abstract

The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function. It is really important to have a definition of the integral that allows a wider class of functions to be integrated. However, there are many other types of integrals, the most important of which is the Lebesgue integral. The Lebesgue integral allows one to integrate unbounded or discontinuous functions whose Riemann integral does not exist, and it has mathematical properties that the Riemann integral does not. The definition of the Lebesgue integral requires the use of measure theory since picking out a suitable class of measurable subsets is an essential prerequisite for Lebesgue integral. The central concepts in this paper are Lebesgue measure and the Lebesgue integral. Examples as well as theorems and proofs will be presented in this paper. In addition, this paper will present some details about the Fundamental Theorem of Calculus for Lebesgue integral.

Dedication

First of all, to all those people who have written the great books and papers that have opened me up to this very enjoyable subject area. There are too many of you for me to list, but in particular I thank C. Ray Rosentrater, Frank Burk, Janovskaja Zermelo, and Steve Cheng. Your wonderful books and clear arguments have made this subject area of joy to learn.

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Finally, and most importantly, to my wonderful family. My husband, my parents, and my little man, my son "Qusai Adi" have been their supporting me through this entire learning process. For this and so much more, I dedicate this thesis to you.

Contents

1 Introduction

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.

– Henri Lebesgue writing to Paul Montel [11].

Henri Lebesgue was born in Beauvais, France, in 1875. The French mathematician Henri Leon Lebesgue developed the Lebesgue integral as a consequence of the problems associated with the Riemann integral. The Lebesgue integral, introduced by Henri Lebesgue in his 1902 dissertation, "Integral, Length, Area", is a generalization of the Riemann integral usually studied in elementary calculus. Lebesgue partitioned the $y - axis$ rather than the $x - axis$ [14]. To see the differing approaches of the two mathematician, imagine that Riemann and Lebesgue were both merchants. Let us say that both of them sold seven items at the following prices: $5, 10, 15, 5, 15, 3, 6$. Riemann would total his sales by adding the numbers as they appear $(5 + 10 + 15 + 5 + 15 + 3 + 6)$. On the other hand, Lebesgue would sum them like this $2(5) + 10 + 2(15) + 3 + 6$. Of course, at the end, the values are the same.

Figure 1 shows an interval from a partition of the range of f [10]. The set

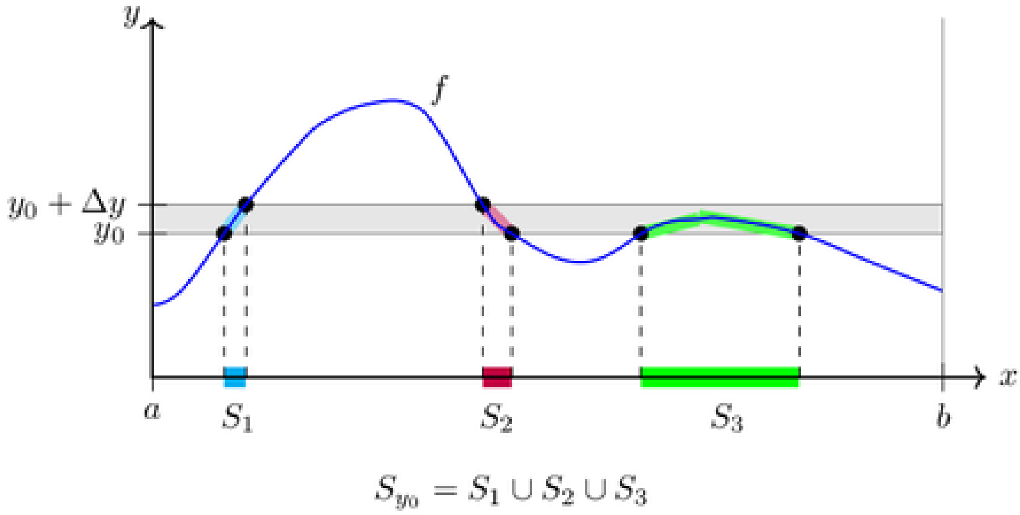

Figure 1: Partitioning the range

of points that f sends into this subinterval is marked on the $x - axis$. The effect of this approach is to gather those points for which f has approximately the same value. When Riemann integral and Lebesgue integral are both defined, they give the same value. There are functions for which the Lebesgue integral is defined but the Riemann integral is not. Thus, the Lebesgue integral is more general than the Riemann integral.

Preimages play a critical role in the Lebesgue integral. Given any set A, the preimage of A under the function f is the set $f^{-1}(A) = \{x \in \Re : f(x) \in A\}$. Suppose we get a function f that is defined on $[a, b]$ and for which $\alpha < f < \beta$. Take any partition $P = \{[y_{k-1}, y_k]\}_{k=1}^{n}$ of $[\alpha, \beta]$ and set $E_k = f^{-1}((y_{k-1}, y_k]), k = 1, 2, ..., n$. Note that $\{E_k\}_{k=1}^{n}$ consists of disjoint sets whose union is all of $[a, b]$.

The lower Lebesgue sum is $S_{\underline{L}}(f,P) = \sum_{k=1}^{n} y_{k-1} * \mu(E_k)$

The upper Lebesgue sum is $S_{\overline{L}}(f,P) = \sum_{k=1}^{n} y_k * \mu(E_k)$

where $\mu(A)$ is the measure of the set A.

Example: The Dirichlet function [6]

Let $f(x)$ be the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

on $[0,1]$. Note that $-1 < f < 2$ and take a partition $P = \{[y_{k-1}, y_k]\}_{k=1}^{n}$ of $[-1,2]$. Now there are unique integers k_0 and k_1 for which $y_{k_0-1} < 0 \leq y_{k_0}$ and $y_{k_1-1} < 1 \leq y_{k_1}$.

Let us follow the definition of the Riemann integral in $\Re$ in order to notice why the Dirichlet function is not a Riemann Integral function.

Riemann: We need to recall the definition of the lower sum, which is the sum of areas $m * \Delta x_k$ using the infimum m of the function $f(x)$ in each subinterval $[x_{k-1}, x_k]$. Also the upper sum is the sum of areas $m * \Delta x_k$ using the supremum m of the function $f(x)$ in each subinterval $[x_{k-1}, x_k]$.

The upper integral is the inf of the "upper sum": $\int_0^{\overline{1}} f(x)dx = 1$

The lower integral is the sup of the "lower sum": $\int_{\underline{0}}^{1} f(x)dx = 0$

since $\int_0^{\overline{1}} f(x)dx \neq \int_{\underline{0}}^{1} f(x)dx$, $f(x)$ is not a Riemann integrable.

The result of this example is that the Riemann integral of a highly discontinuous function does not exist.

Lebesgue:

Let $\mu(E)$ be the Lebesgue measure ("size") of E. Then

$$\int_0^1 f(x)dx = 1 * \mu(\mathbb{Q} \cap [0,1]) + 0 * \mu(\mathbb{Q}^c \cap [0,1])$$

$$= 1 * 0 + 0 * 1 = 0$$

Thus, there are functions that are Lebesgue integrable but not Riemann integralable. Let us now turn to the technical details involved in the Lebesgue integral by developing the "Lebesgue measure" to measure the size of sets. In Lebesgue's theory of integral, we shall see that the fundamental theorem of calculus always holds for any bounded function with an antiderivative [7].

2 Measure and Measurable Sets

A measure $\mu(A)$ is a function that assigns a non-negative real number or ∞ to (certain) subsets of a set X. A measure is a generalization of the concepts of length, area, and volume.

The new "measure" concept should satisfy two conditions

1. It should be applicable to intervals, and unions of intervals. We will satisfy this condition by defining the outer measure.

2. It should share some properties of "length of an interval".

- It is non-negative
- $\mu(A = [a, b]) = b - a$
- Invariant under translation: if $F = E + c = \{e + c \mid e \in E\}$, (in other words, F is a translation by c of the set E), then $\mu(F) = \mu(E)$
- Countably additive: $\mu(A = \cup A_n) = \sum_n (A_n)$, where A_n are pairwise disjoint sets [7].

The definition of the Lebesgue integral thus begins with a measure, μ . In the simplest case, the Lebesgue measure $\mu(A)$ of an interval $A = [a, b]$ is its width, $b - a$, so that the Lebesgue integral agrees with the Riemann integral when both exist. The Riemann integral makes sense only for functions f that are defined on a compact interval, and which are bounded there. Continuous functions are Riemann integrable, and their Riemann and Lebesgue integrals coincide. In more complicated cases, the sets being measured can be highly fragmented, with no continuity and no resemblance to intervals.

2.1 Refinements of partitions

A partition $Q = \{J_1, J_2, ..., J_m\}$ is a refinement of a partition $P = \{I_1, I_2, ..., I_n\}$ if every interval I_k in P is a disjoint union of one or more intervals J_l in Q.

Example: consider the partitions of $[0, 1]$

$$P = \{0, 1/2, 1\}$$

$$Q = \{0, 1/3, 2/3, 1\}$$
$$R = \{0, 1/4, 1/2, 3/4, 1\}.$$

P, Q, and R partition $[0, 1]$ into intervals of equal length $1/2, 1/3$, and $1/4$ respectively. Then Q is not a refinement of P but R is a refinement of P. Consider $S = P \cup Q$ where $S = \{0, 1/3, 1/2, 2/3, 1\}$ is a refinement of both P and Q.

We need a powerful definition to verify that sets are measurable. Therefore, the concept of sigma algebra is a powerful tool for verifying measurability. I will assume that I will deal with sigma algebras for which X is either $\Re$ or a subset of $\Re$.

2.2 Sigma Algebra

Definition Let X be a set. A collection A of subsets of X is a sigma algebra if

1. The empty set belongs to A.
2. If $a \in A$, then the complement of a is in A
3. Given a countable collection of sets a_k from A, we have $\cup_k \ a_k \in A$. Note that the second and third property also implies that $\cap_k \ a_k \in A$ [3].

The pair (X, A) of a non-empty set X and a σ- algebra A of subsets of X is called a measurable space.

Example: Let $A = \{1, 2, 3\}$. Is the trivial set $T = \{\emptyset, A\}$ a σ - algebra?

We need to check if T satisfies the three properties of sigma algebra.

1- The empty set belongs to T.

2- $A^c = \emptyset$, and it is in T.

3- We need to find the union and intersection of sets here and check if each of them belongs to set T. $\emptyset \cup A = A$, and $A \cap \emptyset = \emptyset$.

Therefore our set of subsets T met all the three criterion, so it is a sigma-algebra.

The trivial σ- algebra (also called the minimal σ-algebra) has the least number of elements.

If you have a measurable space (Y, γ), some set X and a family of functions $f_i : X \to Y, i \in I$ for some index set I, then the σ- algebra generated by these functions on X is the σ- algebra generated by the inverse images of the measurable sets in Y. We will note the usefulness of knowing that the collection of measurable sets is a sigma algebra later in verifying the measurability of a function.

Sometimes we refer to the collection of all open sets, closed sets, and sets that can be derived from them as Borel sets. Its significance for our discussion is that all Borel sets are measurable. So, instead of listing open sets, closed sets, unions of them, intersections of them, etc. we refer to them collectively as Borel sets.

Definition. The smallest σ-algebra of subsets of $\Re$ containing all open sets is called the collection β of *Borel sets*.

3 Lebesgue Measure

The pair (X, M) is called a **measurable space**, and the sets in M are called the **measurable sets**. Let us have (X, M) as a measurable space, where M is a $\sigma -$ **algebra**. A measure on this space is a non-negative set function $\mu : M \to [0, \infty)$ such that

1. $\mu(\phi) = 0$ and $\mu([a, b]) = b - a$

2. Non-negativity: $\mu(A) \geq 0, \forall A \in M$

3. Countable additivity: For any sequence of mutually disjoint sets $A_n \in M$

$$\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

.

These are the general conditions, and they form the basis of the field of measure theory.

The goal here is to construct the Lebesgue measure. Let us start with the outer measure of a set. Let $A \subseteq [a, b]$ for some finite $a \neq b \in \Re$ and G is the disjoint union of intervals in $[a, b]$, $G = \cup_i (a_i, b_i)$.

The outer measure $\mu^*(A)$:

$$\mu^*(A) = \inf \{\textstyle\sum_{k=1}^{\infty} \mu(I_k)\}$$

Where the infimum extended over all sequences (I_k) of cells in $\Re$ that cover A in the sense that $A \subseteq \cup_{k=1}^{\infty} I_k$. It is certainly possible to have $\mu^*(A) = \infty$;

for example, this is the case when we take $A = \Re$.

The inner measure $\mu_*(A)$:

$$\mu_*(A) = b - a - \mu^*([a, b] \setminus A).$$

If $\mu_*(A) = \mu^*(A)$, then we say that A is *Lebesgue measurable*.

Example: The following sets are measurable:

* $[a, \infty)$

* $a = \cap_n (a - 1/n \ , \ a + 1/n)$

* $(-\infty, a)$

The second step after defining the outer measure is to relate it to the length of an interval. Outer measure has the following properties:

1. Outer measure μ^* is a non-negative set function whose domain is the power set of $\Re$.

2. The outer measure of an interval is its length [4].

3. Outer measure is subadditive, i.e. $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$, where A and B are mutually disjoint sets.

The set $(X,\ M,\ \mu)$ will be called the **Lebesgue measure space**, which is a triple $(X,\ M,\ \mu)$ consisting of a set X, a σ-algebra M of subsets of X, and a measure μ defined on M.

We need to note that to get from outer measure to Lebesgue measure you need to go through Caratheodory's measurability criterion. Outer measure is countably subadditive but is not countably additive, and what the Caratheodory criterion says is that a set is measurable if and only if it can be used to split any set A into two disjoint pieces for which outer measure is additive.

Definition (Caratheodory's Measurability Criterion) A set of real numbers E is Lebesgue measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$$

holds for every set of real numbers A, and sets E will be called Lebesgue measurable sets.

So we have a criterion for measurability: Select a set E, and check whether it "splits" every set A of real numbers in an additive fashion. If the answer is yes, keep E. Otherwise, discard E [3].

There is a terminological matter that needs to be mentioned and shall be used frequently in Lebesgue measure. we shall say a certain proposition holds μ **- almost everywhere** if there exists a subset $A \in X$ with $\mu(A) = 0$ such that the proposition holds on the complement of A. In this case we will often write it as μ- *a. e.* [2].

A continuous function from $\Re$ to $\Re$ guarantees that for each open interval in its range there is a corresponding open interval in its domain. To see this we need to apply the definition of continuity. For each $a \in \Re$ and for $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x) \in (f(a) - \varepsilon\ ,\ f(a) + \varepsilon)$ for each $x \in (a - \delta\ ,\ a + \delta)$. Now since each Borel set can be written as the countable union of open intervals then for each Borel set in the range of a continuous function the inverse mapping of that set is a countable union of open intervals and thus a Borel set [5].

Examples

* If $A = \{a\}$ is a single point, then $\mu(A) = 0$.

* If $A = \{a_1, a_2, ...\}$ is countable then A is measurable, and $\mu(A) \leq \sum_{i=1}^{\infty} \mu(\{a_i\}) = 0$, so $\mu(A) = 0$. For example $\mu(\mathbb{Q}) = 0$ since $\mu(\mathbb{Q}) = \mu(\cup_{i=1}^{\propto}\{a_i\}) = \sum_{i=1}^{\propto} \mu(\{a_i\}) = \sum_{i=1}^{\propto} 0 = 0$.

* $\mu(\Re^n) = \infty$.

* Every nonempty open set is a finite or countable disjoint union of open intervals $A = \cup(a_i, b_i)$, and $\mu(A) = \sum_i \mu(a_i, b_i) = \sum_i (b_i - a_i)$

4 Measurable Functions

To do the integration theory, we need functions to integrate. We need to note that not all sets may be measurable, it should be expected that not all functions can be integrated either. Measurable functions are candidates for integration. Also, not every function can be integrated by the Lebesgue integral; the function will need to be Lebesgue measurable. Roughly speaking, a function is integrable if its behavior is not too irregular, and if the values which it takes are not too large too often. The second requirement is to have the equality of the upper and lower integrals [13].

If we want to prove that a particular function is measurable, we need to show that the pre-images $f^{-1}((y_{k-1}, y_k])$ are measurable, and this is the critical required condition that makes our computations work.

Sigma algebras prove useful in simplifying this. Since the collection of measurable sets is a sigma algebra and preimages interact well with complements and unions, it suffices to verify that preimages of the form $f^{-1}((-\infty, a))$ are measurable sets.

Definition (Measurable function). Let X be a measurable subset of real numbers and let $f : X \rightarrow \Re$. Then f is **measurable** if the preimage under f of every interval is measurable (Rosentrater 106.)

Example: A constant map f is always measurable, for $f^{-1}(x)$ is either X or ϕ.

Example (Increasing functions are measurable) Suppose X is a measurable set and that $f : X \rightarrow \Re$ is an increasing function. Let $\alpha = sup\{x \in X : f(x) < a\}$. Then $f^{-1}((-\infty, a)) = (-\infty, \alpha) \cap X$ or $f^{-1}((-\infty, a))$ $= (-\infty, \alpha] \cap X$. In either case the preimage of $(-\infty, a)$ is the intersection of two measurable sets and so is measurable.

Example (Continuous functions are measurable) Suppose X is a measurable set and that $f : X \rightarrow \Re$ is continuous. Since $(-\infty, \alpha)$ is open, its preimage under the continuous function f is relatively open in X. In other words, $f^{-1}((-\infty, a))$ is the intersection of X with an open set. Since open sets are measurable, $f^{-1}((-\infty, a))$ is a measurable set. Thus f is a measurable function.

From now on, when there is a given function f, I implicitly assume that it is *measurable*.

5 Lebesgue Integral

Let f be a (Lebesgue) measurable function on $[a, b]$ satisfying $\alpha < f < \beta < \infty$. Let $P = \{[y_{k-1}, y_k]\}_{k=1}^n$ be a partition of $[\alpha, \beta]$ and set $E_k = f^{-1}((y_{k-1}, y_k]), k = 1, 2, ..., n$. The (lower) Lebesgue sum of f with respect to P is

$$S_{\underline{L}}(f, P) = \sum_{k=1}^n y_{k-1} * \mu(E_k)$$

where $\mu(A)$ is the Lebesgue measure of the set A. The Lebesgue integral of f over $[a, b]$ is

$$\int_a^b f d\mu = sup S_{\underline{L}}(f, P)$$

where the supremum is taken over all partitions of $[\alpha, \beta]$. The function f is said to be **Lebesgue integrable** if $\int_a^b f < \infty$. The Lebesgue integral of f is denoted by $\int_a^b f d\mu = {}^L \int_a^b f d\mu$.

There are some properties of the Lebesgue integral. Let f be an integrable function, and A, B denote measurable sets of finite measure

1. If f is a bounded and measurable function, then the integral exists.

2. Integration is a linear operation. That is, if f and g are both integrable and $a, b \in \Re$, then $\int_A (af + bg) d\mu = a \int_A f d\mu + b \int_A g d\mu$.

3. If $f = g$ almost everywhere, then $\int_A f d\mu = \int_A g d\mu$.

4. If $\int_A f(x)^2 dx$ and $\int_A g(x)^2 dx$ both exists, then $\int_A (f(x)+g(x))^2 dx$ exists.

5. If the Riemann integral of f exists, then the Lebesgue integral exists and the integrals are equal [8].

   Example: Lebesgue integral of a step function:

   Let $f(x)$ be a step function defined as:

$$f(x) = \begin{cases} 1, & -1 < x < 2 \\ 2, & 2 \le x < 4 \\ 3, & 4 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

   The Lebesgue integral of $f(x)$ is

   $\int f(x)dx = 1\mu([-1,2]) + 2\mu([2,4]) + 3\mu([4,8]) + 0 = 19$

6 The Fundamental Theorem for the Lebesgue Integral

Theorem (FTC-1) If F is a differentiable function, and the derivative $F^{'}$ is bounded on the interval $[a, b]$, then $F^{'}$ is Lebesgue integrable on

$[a, b]$ and

$$\int_a^x F' d\mu = F(x) - F(a)$$

for x in the interval $[a, b]$.

Proof. First note that F being continuous means it is measurable. Let us assume that F' is bounded. I will extend F to $[a, b+1]$ by setting $F(x) = F(b) + F'(b)(x - b)$ for $x \in [b, b+1]$. Then

$$f_n(x) = n[F(x + 1/n) - F(x)], x \in [a, b]$$

is also measurable with $\{f_n\}$ converging pointwise to F' on $[a, b]$. Hence F' is measurable and is Lebesgue integrable.

By the mean value theorem, given $t \in [a, b]$ and $n \in \mathbb{N}$, there is a value $c \in (t, t + 1/n)$ such that

$$n[F(t + 1/n) - F(t)] = F'(c)$$

As we are assuming that F' is bounded, we see that $\{f_n\}$ is uniformly bounded, which means there is one constant that bounds all functions. Thus we can use the bounded convergence theorem, the continuity of F, and FTC-2 for the Riemann integral (see appendix) to conclude that

$$^L\int_a^x F' d\mu = {}^L\int_a^x \lim \text{ n } [F(t + 1/n) - F(t)]dt$$

$$= \lim \text{n} \ . \ {}^{L}\int_a^x [F(t+1/n) - F(t)]dt$$

$$=\lim \text{n} \ . \ [{}^{L}\int_a^x F(t+1/n)dt - {}^{L}\int_a^x F(t)dt]$$

$$= \lim \text{n} \ . \ [{}^{L}\int_{a+1/n}^{x+1/n} F(t)dt - {}^{L}\int_a^x F(t)dt]$$

$$= \lim \ [n \ . \ {}^{L}\int_x^{x+1/n} F - n \ . \ {}^{L}\int_a^{a+1/n} F]$$

$$= \lim \text{n} \ . \ {}^{R}\int_x^{x+1/n} F \text{ - } \lim \text{n} \ . \ {}^{R}\int_a^{a+1/n} F$$

$$= F(x) - F(a)$$

The proof is complete □.

Notice that it is not possible to prove $\int_a^x F' d\mu = F(x) - F(a)$ without some assumption on F [1].

Comparing this theorem to the corresponding theorem for the Riemann integral, both of them require F to be differentiable on all of $[a, b]$. For the Riemann integral, the FTC requires that F' be continuous whereas the Lebesgue version requires only that F' must be bounded.

Theorem (FTC-2, Lebesgue, 1904). Given that f is Lebesgue integrable on $[a, b]$, define a function f on $[a, b]$ by $F(x) = \int_a^x f d\mu$. Then F is absolutely continuous on $[a, b]$ and $F' = f$ almost everywhere on $[a, b]$.

Vito Vitali Definition: Absolutely Continuous Function. A function f on $[a, b]$ is said to be absolutely continuous on $[a, b]$ if, given any $\epsilon > 0$, we can find a positive number δ such that for any finite collection of pairwise disjoint intervals $(a_k, b_k) \subset [a, b], k = 1, 2, ..., n$, with $\sum(b_k - a_k) < \delta$, we have $\sum |f(b_k) - f(a_k)| < \epsilon$.

Notice how it differs from the FTC-2 for the Riemann integral. First, Riemann integrands must be bounded, and this is no longer a requirement for Lebesgue integrals. Also, Lebesgue's theorem draws the stronger conclusion that F is absolutely continuous. In addition, FTC-2 for the Riemann integral only concludes that $F^{'}(x) = f(x)$ for points where f is continuous. For some Lebesgue integrable such as the Dirichlet function, the set of points of continuity is empty.

Example Let $f(x) = x^2$. the Lebesgue integral of $f(x)$ on $[0, 1]$ is $1/3$, and can be solved by using different methods.

1- By breaking up the range into sub-intervals of length $1/n$ $y_i = i/n, i = 0, 1, ..., n$

$f^{-1}([y_i, y_{i+1}]) = f^{-1}([i/n, i + 1/n])$

Now pick $y_i^* = y_i = i/n$

$\sum_{i=0}^{n}(\sqrt{(i+1)/n} - \sqrt{i/n}) * i/n$

$lim_{n\to\infty} \sum_{i=0}^{n}(\sqrt{(i+1)/n} - \sqrt{i/n}) * i/n = 1/3$

2- Or by using theorem 2,

$\int_0^1 x^2 d\mu = \int_0^1 (x^3/3)^{'} d\mu = 1/3 - 0/3 = 1/3$

Generally speaking, Lebesgue's integral is constructed using a generalization of length called the *measure* of a set. The Lebesgue integral is presently the standard integral in advanced mathematics. The theory is taught to all graduate students, as well as to many advanced undergraduates, and it is the integral used in most research papers where integration is required. The Lebesgue integral generalizes the Riemann integral in the sense that any function that is Riemann integrable is Lebesgue integrable and integrates to the same value. The real strength of the Lebesgue integral is that the class of integrable functions is much larger.

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Appendix

FTC-2 for the Riemann integral. Suppose that f is Riemann integrable on $[a, b]$. Define F on $[a, b]$ by $F(x) =^R \int_a^x f$.

1. Then F is continuous on $[a, b]$

2. If f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 and $F^{'}(x_0) = f(x_0)$.

Mean Value Theorem.if $f(x)$ is defined and continuous on the interval $[a, b]$ and differentiable on (a, b), then there is at least one number c in the interval (a, b) (that is $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Bounded Convergence Theorem. If $\{f_n\}$ is a uniformly bounded sequence of Lebesgue measurable functions converging pointwise to f almost everywhere on $[a, b]$, then

$$\lim \mathrm{L} \int_a^b f_n \; \mathrm{d}\mu = \mathrm{L} \int_a^b f \; \mathrm{d}\mu = \mathrm{L} \int_a^b (\lim \; f_n) \; \mathrm{d}\mu.$$